Accepted Manuscript

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PII: S0305-0483(16)30773-3
DOI: 10.1016/j.omega.2017.05.005
Reference: OME 1780

To appear in: *Omega*

Received date: 15 October 2016
Revised date: 27 April 2017
Accepted date: 5 May 2017

Please cite this article as: Mehdi Toloo , Esmaeil Keshavarz , Adel Hatami-Marbini , Dual-Role Factors for Imprecise Data Envelopment Analysis, *Omega* (2017), doi: 10.1016/j.omega.2017.05.005

Highlights

- A pair of interval DEA models based on the pessimistic and optimistic standpoints is suggested with the aim of dealing with interval dual-role factors.
- A new model is presented which integrates both pessimistic and optimistic models to identify a unique status of each imprecise dual-role factor.
- An aggregate model is formulated with the aim of reallocating the dual-role factors across all DMUs.
- The fuzzy decision-making approach is employed where the interval efficiency score is used to define the fuzzy goal for each DMU.
- A data set of 20 banks is utilized to showcase the applicability and efficacy of the proposed models.

Dual-Role Factors for Imprecise Data Envelopment Analysis

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Abstract

Efficiency analyses are crucial to managerial competency for evaluating the degree to which resources are consumed in the production process of gaining desired services or products. Among the vast available literature on performance analysis, Data Envelopment Analysis (DEA) has become a popular and practical approach for assessing the relative efficiency of Decision-Making Units (DMUs) which employ multiple inputs to produce multiple outputs. However, in addition to inputs and outputs, some situations might include certain factors to simultaneously play the role of both inputs and outputs. Contrary to conventional DEA models which account for precise values for inputs, outputs and dual-role factors, we develop a methodology for quantitatively handling imprecision and uncertainty where a degree of imprecision is not trivial to be ignored in efficiency analysis. In this regard, we first construct a pair of interval DEA models based on the pessimistic and optimistic standpoints to measure the interval efficiencies where some or all observed inputs, outputs and dual-role factors are assumed to be characterized by interval measures. The optimal multipliers associated with the dual-role factors are then used to determine whether a factor is designated as an output, an input, or is in equilibrium even though the status of the dual-role factors may not be unique based upon the pessimistic and optimistic standpoints. To deal with the problem, we present a new model which integrates both pessimistic and optimistic models. The integrated model enables us to identify a unique status of each imprecise dual-role factor as well as to develop a structure for calculating an optimal reallocation model of each dual-role factor among the DMUs. As another method to investigate the role for dual-role factors, we introduce a fuzzy decision-making model which evaluates all DMUs simultaneously. We finally present an application to a data set of 20 banks

to showcase the applicability and efficacy of the proposed procedures and algorithm.

Keywords: Efficiency evaluation; Imprecise data; Dual-role factors; Fuzzy decision-making; Bank industry.

1 Introduction

Data envelopment analysis (DEA) is a popular non-parametric method for measuring the efficiency of a set of Decision-Making Units (DMUs), initially introduced by Farrell (1957) and Charnes et al. (1978). DEA has the capability to compute the relative efficiency of a DMU which is equal to maximizing a ratio of weighted outputs to weighted inputs, subject to the condition that the efficiency score must be less than or equal to one. Since 1980s, the unique characteristics and strength of DEA models lead to its rapid growth and penetration in many fields such as management science, applied mathematics, industrial engineering, economics and so on (see e.g. Emrouznejad et al. (2008) as the comprehensive literature review).

Identifying the given role of each evaluation factor, either input or output role, is one of the premises in conventional DEA models while there are some model structures in which assessment factors could be both inputs and outputs depending on perspective and purpose. The latter factors which can play both input and output roles are known as “dual-role factors”. For instances, “the number of customers” in the evaluation of bank branch performance and “the number of doctoral students”, “research income” or “the number of scholars” in the performance evaluation of university departments can be regarded as the dual-role factor with the simultaneous input and output roles (Cook et al. 2006).

Although Beasley (1990, 1995) was the first who took care of dual-role factors through the evaluation of research productivity of universities, the major attention to dual-role factors centres around its extension model proposed by Cook et al. (2006) where their model takes the weights of dual-role factors into account so as to identify whether a factor is acting like input, output or even has no involvement in the efficiency measure.

In recent years, several studies have been carried out in the DEA literature to deal with dual-role factors, aiming at reflecting the existing complexity and valuation difficulties of a real-world problems (Saen 2010a, 2010b). Generally, the existing approaches for evaluating the performance of firms in the presence of dual-role factors are classified into exogenous and endogenous categories. The exogenous category at first exploits some criteria to identify the classification of a dual-role factor as either an input or an output (Ding et al. 2015). The endogenous category considers a flexibility in determining the classification of a dual-role factor in the DEA model (e.g., Beasley 1990, 1990; Cook et al. 2006; Saen 2010a; Toloo and Barat 2015).

Saen (2010a) proposed a model for seeking the appropriate suppliers in situations where dual-role factors and decision maker's preferences are simultaneously included in the analysis. Given the presence of dual-role factors, Lee and Saen (2012) modified the cross-efficiency model to measure the efficiency of firms from the sustainability perspective. Kumar et al. (2014) slightly modified Saen (2010a)'s model to propose a Green DEA framework for the purpose of formulating supplier selection problems with carbon footprints of suppliers as a dual role factor. Ding et al. (2015) developed a two-step approach to pinpoint the appropriate suppliers in the presence of dual-role factors.

The idea behind DEA is to allow the data to represent themselves, rather than arbitrarily specified functional form. Consequently, the success of DEA directly hinges on the quality and quantity of data. All input and output data for the conventional DEA models are assumed to have the form of crisp numerical values. However, the observed values of the input and output data in real-world problems are at times imprecise and vague, which might be the result of measurement errors, unquantifiable and non-obtainable information. Many research studies are currently conducted for the purpose of improving robustness of DEA by dealing with the imprecise data (e.g., Zhu 2003, Toloo et al. 2008, Toloo 2014).

The existing DEA literature has often acknowledged the necessity of incorporating uncertain, imprecise or vague data such as linguistic, interval, ordinal, and stochastic data within the deterministic DEA models. Stochastic, fuzzy and interval DEA can be recognized as three major categories to overcome uncertainty and obtain robust efficiency measures. In the stochastic DEA category, statistical properties in the production function can be well incorporated through bootstrapping for DEA estimators (Simar and Wilson 1998). Although the use of the bootstrap in the DEA models is a progressively popular practice in many applications such as energy efficiency (e.g., Song et al. 2013), maritime efficiency (e.g., Gutiérrez et al. 2014), health care efficiency (e.g., Shwartz et al. 2016) and so on, the dearth of any particular statistical distribution limits the inferences derived from bootstrapping methods because validity and precision of the bootstrap method hinges on the sample size and the particular dataset. Another way in this category to dealing with data uncertainty includes stochastic non-parametric frontier models, known as Chance Constrained DEA (CCDEA), in which part or all of the input and output values are characterized by the probability distribution (Olesen and Petersen 1995). A foremost shortcoming of the existing stochastic DEA models is to be presumed the unrealistic normal distribution with known means and variances for random data.

In fuzzy DEA category, the implementation of fuzzy sets theory proposed by Zadeh (1965) in DEA has been attracted significant attention to handle the ambiguity and vagueness inherent in evaluation problems. Hatami-Marbini et al. (2011) and Emrouznejad et al. (2014) are two comprehensive surveys of fuzzy DEA methods, categorizing six groups in the literature viz. (1) the tolerance approach (e.g. Sengupta 1992), (2) the α-level based approach (e.g. Hatami-Marbini et al. 2009, 2011, 2013), (3) the fuzzy ranking approach (e.g., Guo and Tanaka 2001; Emrouznejad et al. 2011), (4) the possibility approach (Lertworasirikul et al. 2003) (5) the fuzzy arithmetic (e.g., Hatami-Marbini et al. 2015), and (6) the fuzzy random/type-2 (e.g., Tavana et al. 2012, 2014).

The last decade has witnessed a great increase in the interest for the interval DEA category initiated by Cooper et al. (1999). Theoretically, a pair of DEA models are first constructed under interval input and output data to compute the upper and lower bounds of efficiency from the optimistic and pessimistic viewpoints, respectively, and then the DMUs can be classified into various partitions according to the attained interval efficiencies (e.g., Despotis and Smirlis 2002; Smirlis et al. 2006; Emrouznejad et al. 2012; Hatami-Marbini et al. 2014).

There is a connection between the α-level based approach of fuzzy DEA category and interval DEA category since α-level based approach generally converts the fuzzy DEA model into a pair of parametric models for calculating the lower and upper bounds of the efficiency at a given α- that is to say, interval DEA models can be a special case of the α-level based approach of fuzzy DEA models for a given α.

Mirhedayatian et al. (2014) proposed a network DEA model based on the slacks-based measure (Tone and Tsutsui 2009) for evaluating green supply chain management in the presence of dual-role factors, undesirable outputs, and fuzzy data. The approach to solving their fuzzy network DEA model was based on a fuzzy arithmetic approach in order to obtain fuzzy efficiency scores, which were defuzzified using the concept of truth function (Zimmermann 2013). Sadeghi et al. (2012) firstly took account of the imprecise version of Cook et al. (2006)'s model where input-output data and dual-role factors are characterized by fuzzy number, and then a fuzzy model was defuzzified using the ranking fuzzy numbers method introduced by Abbasbandy and Asady (2006). The focus of Saen (2011) is on the models of Cook et al. (2006) and Cook and Zhu (2007) in the imprecise environment, where he took into account the models in Wang et al. (2005) and weight restrictions in Zhu (2003) to deal with imprecise data.

Referring to the above-mentioned investigation, the consideration of uncertainty embedded in the data and dual-role factors limits to few studies in the DEA literature which bespeaks the insufficient concentration of researchers and scholars in the field. Therefore, we concentrate on the DEA approach with interval inputs, outputs, and dual-role factors. The contribution of this paper is six fold: (1) we develop a pair of DEA models in both the envelopment and multiplier forms to measure interval efficiencies, in which the observed values of inputs, outputs and dual-role factors are varied in their ranges; (2) we focus on a pair of Mixed Integer Linear Programming (MILP) problems to measure the efficiency scores of DMUs from the pessimistic and optimistic viewpoints where the aim is to determine the actual lower and upper bounds of the efficiency for each DMU; (3) we integrate the MILP models in a unified framework to specify the identical status of each dual-role factor in an uncertain environment; (4) based upon an idea originally proposed by Cook et al. (2006), we formulate an imprecise DEA-based model to obtain an optimal reallocation of each dual-role factor equitably allocated across the DMUs; (5) we alternatively introduce a new approach based on the fuzzy decision-making methodology to evaluate all DMUs simultaneously as well as to guarantee a unique role for each dual-role factor; and (6) an application to 20 banks is presented to illustrate the applicability of the proposed procedures and algorithm.

The outline of this paper is as follows: Section 2 presents an overview of DEA models without and with dual-role factors, and imprecise DEA models. In Section 3, the detailed DEA-based models proposed in this study are introduced. Section 4 illustrates the applicability of the proposed models and methodology by using a real data set of banking industry. Conclusion and further research directions are discussed in Section 5.

2 Background

In this section, we provide a brief overview of three distinct DEA models under constant returns to scale (CRS) assumption. The first model is the primary CCR model with precise input and output data developed by Charnes et al. (1978). The second one presents an extension of the CCR model, considering dual-role factors with precise data, and the last model refers a CCR model with interval inputs and outputs with the aim of finding an efficiency interval for each DMU.

2.1 DEA models

In DEA, the efficiency score of a DMU is defined as the maximum ratio of the weighted sum of outputs to the weighted sum of inputs subject to the condition that the corresponding ratio for each DMU is less than or equal to one and all input and output weights are nonnegative.

Consider n DMUs to be evaluated. DMU_j $(j = 1, \dots, n)$ consumes a semi-positive input vector $\boldsymbol{x}_j = (x_{1j}, \dots, x_{mj})$ to produce a semi-positive output vector $\boldsymbol{y}_j = (y_{1j}, \dots, y_{sj})$. The input-oriented CCR model for a particular DMU_o under evaluation can be expressed using the following linear programming (LP) model (Charnes et al. 1978):

$$
\begin{aligned}
& e_o = \max \textstyle\sum_{r=1}^{s} u_r y_{ro} \\
& \text{s.t.} \\
& \textstyle\sum_{i=1}^{m} v_i x_{io} = 1 \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad && \forall j \\
& u_r, v_i \ge 0 && \forall r, i
\end{aligned}
\tag{1}
$$

where u_r and v_i are the output and input weights, respectively, and e_o presents the efficiency score of DMU_o. Let the optimal solution of model (1) be $(\boldsymbol{u}^*, \boldsymbol{v}^*) = (u_1^*, \dots, u_s^*, v_1^*, \dots, v_m^*)$. DMU_o is said to be *CCR-efficient* if $e_o = 1$ and there exists at least a strictly positive optimal solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$, otherwise, DMU_o is said to be *CCR-inefficient* (Cooper et al. 2007). Model (1) is known as *multiplier* form of CCR model and its dual is called *envelopment* form of CCR model as presented below:

$$
\begin{aligned}
& e_o = \min \theta \\
& \text{s.t.} \\
& \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} \quad && \forall i \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && \forall r \\
& \lambda_j \ge 0 && \forall j
\end{aligned}
\tag{2}
$$

where λ_j $(j = 1, \dots, n)$ is the intensity variable associated with the j^{th} DMU for connecting the inputs and outputs. The objective of the above model is to estimate the production frontier by seeking the maximum possible proportional reduction in all inputs while output levels of each DMU held constant.

2.2 Dual-role factors

In conventional DEA models, each factor plays the role of an input or output, even though in some cases a factor, known as *dual-role factor*, can be simultaneously considered as both input and output. Assume that there are K dual-role factors, denoted by w_k $(k = 1, \ldots, K)$. Cook et al. (2006) proposed the following model to accommodate dual-role factors in model (1):

$$\begin{aligned}
&\max \textstyle\sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{K} \gamma_k w_{ko} - \sum_{k=1}^{K} \delta_k w_{ko} \\
&\text{s.t.} \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} = 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{K} \gamma_k w_{kj} - \sum_{k=1}^{K} \delta_k w_{kj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad \forall j \\
&u_r, v_i, \gamma_k, \delta_k \geq 0 \quad \forall r, i, k
\end{aligned} \tag{3}$$

It is assumed that each dual-role factor w_k serves as both an input and output factor where each dual-role variable is considered as a non-discretionary factor. Importantly, since the difference associated with the coefficient vectors of variables γ_k and δ_k, i.e. $(w_{k1}, \ldots w_{kn})^T$ and $(-w_{k1}, \ldots - w_{kn})^T$, is attributed to their sign, these variables are dependent and cannot enter to basic feasible solution at the same time. As a result, at optimality, due to $\gamma_k^* \delta_k^* = 0$ for $k = 1, \ldots, K$, one of the following three cases may occur:

1. If $\boldsymbol{\delta_k^*} > \mathbf{0}$, then the dual-role factor $\boldsymbol{w_k}$ plays the role of a non-discretionary input.
2. If $\boldsymbol{\gamma_k^*} > \mathbf{0}$, then the dual-role factor $\boldsymbol{w_k}$ plays the role of an output.
3. If $\boldsymbol{\delta_k^*} = \boldsymbol{\gamma_k^*} = \mathbf{0}$, then the dual-role facto $\boldsymbol{w_k}$ is neglected.

2.3 Interval data

A factor is characterized by the interval measure when its exact value is unknown but its true value varies within a bounded interval. In line with the notations defined earlier, let us assume that $x_{ij} \in [x_{ij}^l, x_{ij}^u]$ and $y_{rj} \in [y_{rj}^l, y_{rj}^u]$ where $x_{ij}^l \geq 0$ and $y_{rj}^l \geq 0$. To deal with interval data in the DEA model (1), Despotis and Smirlis (2002) extended the following pair of models to compute the upper and lower bounds of the efficiency score for DMU_o:

$$
\begin{aligned}
& e_o^u = \max \textstyle\sum_{r=1}^{s} u_r y_{ro}^u \\
& \text{s.t.} \\
& \textstyle\sum_{i=1}^{m} v_i x_{io}^l = 1 \\
& \textstyle\sum_{r=1}^{s} u_r y_{ro}^u - \sum_{i=1}^{m} v_i x_{io}^l \leq 0 \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj}^l - \sum_{i=1}^{m} v_i x_{ij}^u \leq 0 \quad \forall j \neq o \\
& u_r, v_i \geq 0 \quad \forall r, i
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
& e_o^l = \max \textstyle\sum_{r=1}^{s} u_r y_{ro}^l \\
& \text{s.t.} \\
& \textstyle\sum_{i=1}^{m} v_i x_{io}^u = 1 \\
& \textstyle\sum_{r=1}^{s} u_r y_{ro}^l - \sum_{i=1}^{m} v_i x_{io}^u \leq 0 \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 \quad \forall j \neq o \\
& u_r, v_i \geq 0 \quad \forall r, i
\end{aligned}
\tag{5}
$$

where $e_o^l \leq e_o^u$. As a matter of fact, models (4) and (5) include two different scenarios for finding lower and upper bounds of efficiency scores. Model (4) is concentrating on the optimistic scenario for an evaluated unit to calculate the upper bound of efficiency scores, in which DMU_o thinks of the lower bounds for inputs and the upper bounds for outputs, and the technology (alternatively called production possibly set) is constructed by the lower bounds for outputs and the upper bounds for inputs. To get the lower bound of the efficiency score, model (5) considers the pessimistic scenario for an evaluated unit where DMU_o is built based on the upper bounds for inputs and the lower bounds for outputs, and the technology is constructed by the upper bounds for outputs and the lower bounds for inputs.

After a careful scrutiny of the above models, we observe a sharp distinction between models (4) and (5). The former takes accounts of the best situation for DMU_o and the worst situation for DMU_j ($\forall j \neq o$) to estimate the production frontier and the latter consists of the worst situation for DMU_o and the best situation for DMU_j ($\forall j \neq o$).

3 Proposed method

Roughly speaking, uncertain data in DEA models can be presented from three different points of view: stochastic, fuzzy and interval data. Here, our main focus in this research is on interval data along with taking into account some interval dual-role factors. In what follows, we first extend a pair of models to determine the upper and lower bounds of the best relative efficiency score of each DMU with interval data. Second, we propose an integrated model to obtain the interval efficiency of a DMU where each dual-role factor is characterized an identical role. Third, a model is formulated to obtain an optimal reallocation of every dual-role factor across the DMUs in situations where DMUs are centralized and decisions are made and resources allocated by a central authority. We lastly apply a fuzzy decision-making methodology where the interval efficiencies obtained from the proposed models are deemed to be fuzzy goals.

3.1 Finding interval efficiencies

Assume that for each DMU_j in addition to m interval inputs $x_{ij} \in [x_{ij}^l, x_{ij}^u], i = 1, \dots, m,$ and s interval outputs $y_{rj} \in [y_{rj}^l, y_{rj}^u], r = 1, \dots, s,$ there are K interval dual-role factors $w_{kj} \in [w_{kj}^l, w_{kj}^u], k = 1, \dots, K,$ which might be input or output. In order to measure the upper bound of the best relative efficiency of DMU_o, we consider dual-role factors besides inputs and outputs by adjusting their levels in favour of DMU_o and aggressively against the other DMUs. That is to say, the dual-role factor of DMU_o, w_{ko}, is defined as the lower bound w_{ko}^l with the input character and the upper bound w_{ko}^u, with the output character whereas the dual-role factor for $\mathrm{DMU}_j, \forall\, j \neq o$, w_{kj}, takes w_{kj}^u for its input role and w_{kj}^l for its output role. It seems that the following LP model enables us to measure the upper bound for the efficiency score e_o^u:

$$
\begin{aligned}
& e_o^u = \max \textstyle\sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \gamma_k w_{ko}^u - \sum_{k=1}^{K} \delta_k w_{ko}^l \\
& \text{s.t.} \\
& \textstyle\sum_{i=1}^{m} v_i x_{io}^l = 1 \\
& \textstyle\sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \gamma_k w_{ko}^u - \sum_{k=1}^{K} \delta_k w_{ko}^l - \sum_{i=1}^{m} v_i x_{io}^l \leq 0 \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj}^l + \sum_{k=1}^{K} \gamma_k w_{kj}^l - \sum_{k=1}^{K} \delta_k w_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^u \leq 0 \qquad \forall j \neq o \\
& u_r, v_i, \gamma_k, \delta_k \geq 0 \qquad \forall r, i, k
\end{aligned}
\tag{6}
$$

while the constraint coefficient vector for variables γ_k and δ_k, i.e. $\left(w_{ko}^u, w_{k1}^l, \ldots, w_{k,o-1}^l, w_{k,o+1}^l, \ldots, w_{kn}^l\right)^T$ and $-\left(w_{ko}^l, w_{k1}^u, \ldots, w_{k,o-1}^u, w_{k,o+1}^u, \ldots, w_{kn}^u\right)^T$, are independent and hence these variables might enter to the basic feasible solution simultaneously, that is, this factor impossibly takes both input and output roles. We deal with the problem by defining auxiliary binary variables (for more details see Williams 2013) to reformulate model (6) as the following Mixed Integer Non-Linear Programming (MINLP) problem:

$$
\begin{array}{ll}
e_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} b_k \gamma_k w_{ko}^u - \sum_{k=1}^{K} d_k \delta_k w_{ko}^l & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io}^l = 1 & \\
\sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} b_k \gamma_k w_{ko}^u - \sum_{k=1}^{K} d_k \delta_k w_{ko}^l - \sum_{i=1}^{m} v_i x_{io}^l \leq 0 & \\
\sum_{r=1}^{s} u_r y_{rj}^l + \sum_{k=1}^{K} b_k \gamma_k w_{kj}^l - \sum_{k=1}^{K} d_k \delta_k w_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^u \leq 0 & \forall j \neq o \\
b_k + d_k \leq 1 & \forall k \\
u_r, v_i, \gamma_k, \delta_k \geq 0 & \forall r, i, k \\
b_k, d_k \in \{0,1\} & \forall k
\end{array}
\tag{7}
$$

where b_k and d_k are the binary variables. Note that the constraint $b_k + d_k \leq 1$ provides a situation where w_k takes at most one role, that is, the dual-role factor w_k is considered as an output or input if $\{b_k = 1, d_k = 0\}$ and $\{b_k = 0, d_k = 1\}$, respectively. In addition, w_k is at the equilibrium status if $\{b_k = 0, d_k = 0\}$. Although model (7) is non-linear due to $b_k\gamma_k$ and $d_k\delta_k$, we replace $b_k\gamma_k$ and $d_k\delta_k$ with $\hat{\gamma}_k$ and $\hat{\delta}_k$, respectively, together with imposing the constraints $0 \leq \hat{\gamma}_k \leq b_k M$ and $0 \leq \hat{\delta}_k \leq M d_k$ to transform model (7) to the following MILP:

$$
\begin{array}{ll}
e_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^u - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^l & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io}^l = 1 & \\
\sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^u - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^l - \sum_{i=1}^{m} v_i x_{io}^l \leq 0 & \\
\sum_{r=1}^{s} u_r y_{rj}^l + \sum_{k=1}^{K} \hat{\gamma}_k w_{kj}^l - \sum_{k=1}^{K} \hat{\delta}_k w_{kj}^u - \sum_{i=1}^{m} v_i x_{ij}^u \leq 0 & \forall j \neq o \\
0 \leq \hat{\gamma}_k \leq M b_k & \forall k \\
0 \leq \hat{\delta}_k \leq M d_k & \forall k \\
b_k + d_k \leq 1 & \forall k \\
u_r, v_i \geq 0 & \forall r, i, k \\
b_k, d_k \in \{0,1\} & \forall k
\end{array}
\tag{8}
$$

where M is a sufficiently large positive number. Likewise, the following model can be developed to measure the lower bound of efficiency e_o^l.

$$
\begin{aligned}
& e_o^l = \max \textstyle\sum_{r=1}^{s} u_r y_{ro}^l + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^l - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^u \\
& \text{s.t.} \\
& \textstyle\sum_{i=1}^{m} v_i x_{io}^u = 1 \\
& \textstyle\sum_{r=1}^{s} u_r y_{ro}^l + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^l - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^u - \sum_{i=1}^{m} v_i x_{io}^u \le 0 \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{K} \hat{\gamma}_k w_{kj}^u - \sum_{k=1}^{K} \hat{\delta}_k w_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^l \le 0 && \forall j \ne o \\
& 0 \le \hat{\gamma}_k \le M b_k && \forall k \\
& 0 \le \hat{\delta}_k \le M d_k && \forall k \\
& b_k + d_k \le 1 && \forall k \\
& u_r, v_i \ge 0 && \forall r, i, k \\
& b_k, d_k \in \{0,1\} && \forall k
\end{aligned}
\tag{9}
$$

Contrary to model (8), the dual-role factors and input and output levels in model (9) are adjusted unfavourably for DMU_o under evaluation and in favour of the other DMUs. In other words, for DMU_o, each dual-role factor takes its upper bound as an input and its lower bound as an output role, and the dual role factor of the remaining DMUs takes account of its lower bound as an input role and its upper bounds as an output role.

Though the main objective of model (9) is to calculate the lowest bound e_o^l for the relative efficiency for DMU_o, we can demonstrate that e_o^l may not be the lowest bound of the interval efficiency and it is possible to find an efficiency for DMU_o that is lower than e_o^l (see Theorem 1). Put differently, the conventional approach developed by Despotis and Smirlis (2002) for computing the lower bound of efficiency interval in model (9) may not be further valid due to auxiliary binary variables. To attain the efficiency lower than e_o^l, it is need to focus on the *envelopment* form of model (9). To this end, similar to model (9), we evaluate DMU_o from the pessimistic viewpoint that considers the worst situation for DMU_o and the best situation for the other DMUs as represented below:

$$
x_{ij} = \begin{cases} x_{ij}^u & j = o \\ x_{ij}^l & j \ne o \end{cases}, \qquad y_{rj} = \begin{cases} y_{rj}^l & j = o \\ y_{rj}^u & j \ne o \end{cases}, \qquad w_{kj} = \begin{cases} \text{input role} \begin{cases} w_{kj}^u & j = o \\ w_{kj}^l & j \ne o \end{cases} \\ \text{output role} \begin{cases} w_{kj}^l & j = o \\ w_{kj}^u & j \ne o \end{cases} \end{cases}
$$

As a result, the i^{th} input-constraint $\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}$ and r^{th} output-constraint $\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}$ in model (2) are transformed into $\sum_{j=1(j\neq o)}^{n} \lambda_j x_{ij}^{l} + \lambda_o x_{io}^{u} \leq \theta x_{io}^{u}$ and $\sum_{j=1(j\neq o)}^{n} \lambda_j y_{rj}^{u} + \lambda_o y_{ro}^{l} \geq y_{ro}^{l}$, respectively. In order to characterize the k^{th} dual-role factor w_k, three different cases may arise: (i) if w_k is considered as a non-discretionary input, then the input-constraint $\sum_{j=1(j\neq o)}^{n} \lambda_j w_{kj}^{l} + \lambda_o w_{ko}^{u} \leq w_{ko}^{u}$ to be satisfied, (ii) if w_k takes the role of an output, then its corresponding constraint is $\sum_{j=1(j\neq o)}^{n} \lambda_j w_{kj}^{u} + \lambda_o w_{ko}^{l} \geq w_{ko}^{l}$, and (iii) the factor w_k possesses neither input nor output role at an equilibrium status, in this case no constraint should be active for the dual-role factor. To handle these three cases, we utilize two auxiliary binary variables b_k and d_k as well as the following three extra constraints (i.e. (10.d), (10.e) and (10.f)) in the corresponding envelopment model to calculate the lowest bound for efficiency score of DMU_o:

$$
\begin{array}{llll}
\bar{e}_o^{l} = \min \theta & & \text{(a)} & \\
\text{s.t.} & & & \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij}^{l} + \lambda_o x_{io}^{u} \leq \theta x_{io}^{u} & \forall i & \text{(b)} & \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj}^{u} + \lambda_o y_{ro}^{l} \geq y_{ro}^{l} & \forall r & \text{(c)} & \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j w_{kj}^{l} + \lambda_o w_{ko}^{u} \leq w_{ko}^{u} + M(1-d_k) & \forall k & \text{(d)} & (10) \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j w_{kj}^{u} + \lambda_o w_{ko}^{l} \geq w_{ko}^{l} - M(1-b_k) & \forall k & \text{(e)} & \\
b_k + d_k \leq 1 & \forall k & \text{(f)} & \\
\lambda_j \geq 0 & \forall j & \text{(g)} & \\
b_k, d_k \in \{0,1\} & \forall k & \text{(h)} &
\end{array}
$$

The (10.d) constraint is replaced by $\sum_{j=1(j\neq o)}^{n} \lambda_j w_{kj}^{l} + \lambda_o w_{ko}^{u} \leq w_{ko}^{u}$ if $d_k = 1$, and the dual-role factor w_k is treated as an input, and if $d_k = 0$, then the constraint $\sum_{j=1(j\neq o)}^{n} \lambda_j w_{kj}^{l} + \lambda_o w_{ko}^{u} \leq w_{ko}^{u} + M$ is redundant. In the same manner, if $b_k = 1$, then the (10.e) constraint is transformed into $\sum_{j=1(j\neq o)}^{n} \lambda_j w_{kj}^{u} + \lambda_o w_{ko}^{l} \geq w_{ko}^{u}$, and w_k is considered as an output, and if $\mathrm{b}_k = 0$, this constraint is redundant. Note that the constraint (10.f) compels w_k to take at most one role at a time, and in case of $b_k = d_k = 0$, the dual role factor w_k is at the equilibrium status.

More importantly, Theorem 1 proves that the optimal objective value of model (9) is an upper bound for the optimal objective value of model (10):

Theorem 1 . $\bar{e}_o^l \leq e_o^l$.

Proof. Let $(\boldsymbol{u}^*, \boldsymbol{v}^*, \widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\delta}}^*, \boldsymbol{b}^*, \boldsymbol{d}^*)$ and $(\theta^*, \boldsymbol{\lambda}^*, \overline{\boldsymbol{b}}^*, \overline{\boldsymbol{d}}^*)$ be the optimal solutions of models (9) and (10), respectively. If one defines the index sets $\overline{K}_b = \{k: \bar{b}_k^* = 1\}$ and $\overline{K}_d = \{k: \bar{d}_k^* = 1\}$; it is clear that $\overline{K}_b \cap \overline{K}_d = \emptyset$ due to the (10.f) constraint. Consider the following LP model:

$$
\begin{aligned}
& \min \theta \\
& \text{s.t.} \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j\, x_{ij}^l + \lambda_o x_{io}^u \leq \theta x_{io}^u && \forall i \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j\, y_{rj}^u + \lambda_o y_{ro}^l \geq y_{ro}^l && \forall s \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j\, w_{kj}^l + \lambda_o w_{ko}^u \leq w_{ko}^u && \forall k \in \overline{K}_d \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j\, w_{kj}^u + \lambda_o w_{ko}^l \geq w_{ko}^l && \forall k \in \overline{K}_b \\
& \lambda_j \geq 0 && \forall j
\end{aligned}
\tag{11}
$$

It is straightforward to verify that the optimal solution of model (11) is $(\theta^*, \boldsymbol{\lambda}^*)$ and its optimal objective value is $\bar{e}_o^l$. The dual problem of model (11) is expressed as follows:

$$
\begin{aligned}
& \max\ \sum_{r=1}^{s} u_r y_{ro}^l + \sum_{k\in\overline{K}_b} \gamma_k w_{ko}^l - \sum_{k\in\overline{K}_d} \delta_k w_{ko}^u \\
& \text{s.t.} \\
& \sum_{i=1}^{m} v_i x_{io}^u = 1 \\
& \sum_{r=1}^{s} u_r y_{ro}^l + \sum_{k\in\overline{K}_b} \gamma_k w_{ko}^l - \sum_{k\in\overline{K}_d} \delta_k w_{ko}^u - \sum_{i=1}^{m} v_i x_{io}^u \leq 0 \\
& \sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k\in\overline{K}_b} \gamma_k w_{kj}^u - \sum_{k\in\overline{K}_d} \delta_k w_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 && \forall j \neq o \\
& u_r, v_i \geq 0 && \forall r, i \\
& \gamma_k \geq 0 && \forall k \in \overline{K}_b \\
& \delta_k \geq 0 && \forall k \in \overline{K}_d
\end{aligned}
\tag{12}
$$

Let $(\boldsymbol{u}', \boldsymbol{v}', \boldsymbol{\gamma}', \boldsymbol{\delta}')$ be the optimal solution of model (12) where $\boldsymbol{\gamma}' = (\dots, \gamma_k', \dots)_{k\in\overline{K}_b}$ and $\boldsymbol{\delta}' = (\dots, \delta_k', \dots)_{k\in\overline{K}_d}$. Given the strong duality theorem (Bazaraa et al. 2010), the optimal objective value of model (12) is $\bar{e}_o^l$.

Let

$$b'_k = \begin{cases} 1 & k \in \bar{K}_b \\ 0 & k \notin \bar{K}_b \end{cases}, \qquad d'_k = \begin{cases} 1 & k \in \bar{K}_d \\ 0 & k \notin \bar{K}_d \end{cases}, \qquad k = 1, \dots, K,$$

$$\hat{\gamma}'_k = \begin{cases} \gamma'_k & k \in \bar{K}_b \\ 0 & k \notin \bar{K}_b \end{cases}, \quad \hat{\delta}'_k = \begin{cases} \delta'_k & k \in \bar{K}_d \\ 0 & k \notin \bar{K}_d \end{cases}, \qquad k = 1, \dots, K.$$

A simple computation clarifies that $(\boldsymbol{u}', \boldsymbol{v}', \hat{\boldsymbol{\gamma}}', \hat{\boldsymbol{\delta}}', \boldsymbol{b}', \boldsymbol{d}')$ is a feasible solution of model (9) with the objective value $\bar{e}_o^l$ and hence $\bar{e}_o^l \leq e_o^l$, which completes the proof. □

The set of interval efficiencies $\{[\bar{e}_j^l, e_j^u], j = 1, \dots, n\}$ is the result of models (8) and (10). However, the status of a dual-role factor, as a purpose of our research, may not be unique on account of two causes. First, models (8) and (10) are independently solved for DMU_o and there is no guarantee to determine a unique designation for each dual-role factor in the lower and upper bound formulations. Second, models (8) or (10) cannot guarantee to yield an identical designation for each dual-role factor over all DMUs. Subsections 3.2 and 3.4 will treat the first and second aforementioned problems and Subsections 3.3 develop an apt model structure for reallocation of a dual-role factor in some situations with a central authority.

3.2 Unique status for a dual-role factor

Though models (8) and (10) have the capability to provide a bounded interval $[\bar{e}_o^l, e_o^u]$ for DMU_o, the status of dual-role factors for this DMU may be different based on this pair of models. This issue can come into question since the decision-makers (DMs) often tend to be aware of the unique designation of each dual-role factor. To treat the problem, we develop an integrated model based on models (8) and (10). Prior to delving into this integrated model, let us integrate models (4) and (5) in the absence of dual-role factors to throw light on the idea used here. Therefore, consider the following integrated model:

$$
\begin{aligned}
&\max \ \textstyle\sum_{r=1}^{s} u_r y_{ro}^{u} - \theta \\
&\text{s.t.} \\
&\textstyle\sum_{i=1}^{m} v_i x_{io}^{l} = 1 \\
&\textstyle\sum_{r=1}^{s} u_r y_{ro}^{u} - \sum_{i=1}^{m} v_i x_{io}^{l} \le 0 \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj}^{l} - \sum_{i=1}^{m} v_i x_{ij}^{u} \le 0 && \forall j \ne o \\
&\textstyle\sum_{\substack{j=1 \\ j\ne o}}^{n} \lambda_j \, x_{ij}^{l} + \lambda_o x_{io}^{u} \le \theta x_{io}^{u} && \forall i \\
&\textstyle\sum_{\substack{j=1 \\ j\ne o}}^{n} \lambda_j \, y_{rj}^{u} + \lambda_o y_{ro}^{l} \ge y_{ro}^{l} && \forall \, r \\
&\lambda_j \ge 0 && \forall j \\
&u_r, v_i \ge 0 && \forall r, i
\end{aligned}
\tag{13}
$$

The constraints of model (13) includes the union of constraints of models (4) and (5). It should be noted that although both lower and upper bounds of inputs and outputs of all DMUs have appeared simultaneously in model (13), each bound is identically and independently used to obtain one of the extreme values of efficiency interval. As such, the following lemma bespeaks the equivalence of model (13) and models (4) and (5).

Lemma 1. $(\boldsymbol{u}^*, \boldsymbol{v}^*, \theta^*, \boldsymbol{\lambda}^*)$ is an optimal solution of model (13) if and only if $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ and $(\theta^*, \boldsymbol{\lambda}^*)$ are the optimal solutions of models (4) and (5), respectively.

Proof. The proof is straightforward and we leave it to the reader. □

The above lemma shows that if $(\boldsymbol{u}^*, \boldsymbol{v}^*, \theta^*, \boldsymbol{\lambda}^*)$ is the optimal solution of model (13), then the lower and upper bounds of efficiency score of DMU_o are θ^* and $\sum_{r=1}^{s} u_r^* y_{ro}^{u}$, respectively.

At present, we propose model (14) as an extended version of model (13) in the presence of dual-role factors, which integrates models (8) and (10).

$$
\begin{array}{ll}
\max\ \left[\sum_{r=1}^{s} u_r y_{ro}^{u} + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^{u} - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^{l}\right] - \theta & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io}^{l} = 1 & \\
\sum_{r=1}^{s} u_r y_{ro}^{u} + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^{u} - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^{l} - \sum_{i=1}^{m} v_i x_{io}^{l} \leq 0 & \\
\sum_{r=1}^{s} u_r y_{rj}^{l} + \sum_{k=1}^{K} \hat{\gamma}_k w_{kj}^{l} - \sum_{k=1}^{K} \hat{\delta}_k w_{kj}^{u} - \sum_{i=1}^{m} v_i x_{ij}^{u} \leq 0 & \forall j \neq o \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij}^{l} + \lambda_o x_{io}^{u} \leq \theta x_{io}^{u} & \forall i \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj}^{u} + \lambda_o y_{ro}^{l} \geq y_{ro}^{l} & \forall r \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j w_{kj}^{l} + \lambda_o w_{ko}^{u} \leq w_{ko}^{u} + M(1-d_k) & \forall k \\
\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j w_{kj}^{u} + \lambda_o w_{ko}^{l} \geq w_{ko}^{l} - M(1-b_k) & \forall k \\
0 \leq \hat{\gamma}_k \leq M b_k & \forall k \\
0 \leq \hat{\delta}_k \leq M d_k & \forall k \\
b_k + d_k \leq 1 & \forall k \\
\lambda_j \geq 0 & \forall j \\
u_r, v_i \geq 0 & \forall r, i, k \\
b_k, d_k \in \{0,1\} & \forall k
\end{array}
\tag{14}
$$

The objective function of model (14) involves two distinct terms. The first term $\sum_{r=1}^{s} u_r y_{ro}^{u} + \sum_{k=1}^{K} \hat{\gamma}_k w_{ko}^{u} - \sum_{k=1}^{K} \hat{\delta}_k w_{ko}^{l}$ and the second term θ are the objective function of models (8) and (10), respectively. The set of variables and constraints of model (14) are the union of variables and constraints of models (8) and (10). The binary variables b_k and d_k play crucial roles in model (14) and they act as a bridge between models (8) and (10). The value of these binary variables may differ when one solves the two independent models (8) and (10), although model (14) enables us to obtain an identical value for each binary variable, resulting in a unique status for each dual-role factor.

The following lemma and theorem showcase the relation between model (14) and both models (8) and (10) in terms of the feasible solutions and the optimal objective values.

Lemma 2. $(\boldsymbol{u}, \boldsymbol{v}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\delta}}, \theta, \boldsymbol{\lambda}, \boldsymbol{b}, \boldsymbol{d})$ is a feasible solution of model (14) if and only if $(\boldsymbol{u}, \boldsymbol{v}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\delta}}, \boldsymbol{b}, \boldsymbol{d})$ and $(\theta, \boldsymbol{\lambda}, \boldsymbol{b}, \boldsymbol{d})$ are the feasible solutions of models (8) and (10), respectively.

Proof. The proof is straightforward. □

Theorem 2 . Let $(\boldsymbol{u}^*, \boldsymbol{v}^*, \widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\delta}}^*, \theta^*, \boldsymbol{\lambda}^*, \boldsymbol{b}^*, \boldsymbol{d}^*)$ be the optimal solution of model (14), $\bar{\xi}_o^l = \theta^*$ and $\xi_o^u = \sum_{r=1}^s u_r^* y_{ro}^u + \sum_{k=1}^K \hat{\gamma}_k^* w_{ko}^u - \sum_{k=1}^K \hat{\delta}_k^* w_{ko}^l$. Then, $[\bar{\xi}_o^l, \xi_o^u] \subseteq [\bar{e}_o^l, e_o^u]$.

Proof. Given Lemma 2, $(\boldsymbol{u}^*, \boldsymbol{v}^*, \widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\delta}}^*, \boldsymbol{b}^*, \boldsymbol{d}^*)$ and $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{b}^*, \boldsymbol{d}^*)$ are the feasible solutions for models (8) and (10), respectively. Therefore, the objective value of model (8) for $(\boldsymbol{u}^*, \boldsymbol{v}^*, \widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\delta}}^*, \boldsymbol{b}^*, \boldsymbol{d}^*)$, i.e. ξ_o^u, is less than or equal to the optimal objective value e_o^u of this model. Likewise, we can show that $\bar{\xi}_o^l \geq \bar{e}_o^l$, which completes the proof. □

According to the above theorem, we can amend the efficiency interval $[\bar{e}_o^l, e_o^u]$ for DMU_o to $[\bar{\xi}_o^l, \xi_o^u]$ where the status of each dual-role factor is unique. In the next section, we utilize the modified efficiency interval $[\bar{\xi}_o^l, \xi_o^u]$ calculated from model (14) so as to propose an aggregate efficiency model that leads to an appropriate structure for reallocation of dual-role factors across DMUs.

3.3 Optimal allocation of a dual-role factor

After accommodating the dual-role factor, Cook et al. (2006) formulated an aggregate model for a situation where a specific dual-role factor is optimally reallocated across all DMUs by a central authority. In this subsection, we develop the model of Cook et al. (2006) under the situation of imprecise data to qualify for reallocation of a dual-role factor .

Assume that W_k^L and $W_k^U, k = 1, \dots, K$, are the minimum and maximum amounts of available resources for the k^{th} dual-role factor, which a DM is inclined to reallocate to all DMUs in a fairly and reasonable manner. Let $\{[\bar{\xi}_j^l, \xi_j^u], j = 1, \dots, n\}$ be the set of optimal interval efficiencies calculated from model (14). Furthermore, let us define the following sets:

$$J_{out}^k = \{j: w_k \text{ plays the role of an output for } \text{DMU}_j\}$$

$$J_{in}^k = \{j: w_k \text{ plays the role of an input for } \text{DMU}_j\}$$

$$\pi_{kj} = \begin{cases} 1 & j \in J_{out}^k \\ -1 & j \in J_{in}^k \\ 0 & otherwise \end{cases}$$

We then propose the following non-linear programming (NLP) problem:

$$\begin{aligned}
&\max \textstyle\sum_{j=1}^{n}\left[\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{K} \pi_{kj}\alpha_k w_{kj}\right] && && \text{(a)} \\
&\text{s.t.} && && \\
&\textstyle\sum_{j=1}^{n}\left[\sum_{i=1}^{m} v_i x_{ij}\right] = 1 && && \text{(b)} \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{K} \pi_{kj}\alpha_k w_{kj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 && \forall j && \text{(c)} \\
&W_k^L \le \textstyle\sum_{j=1}^{n}\left|\pi_{kj}\right| w_{kj} \le W_k^U && \forall k && \text{(d)} \\
&x_{ij}^l \le x_{rj} \le x_{ij}^u && \forall k, j && \text{(e)} \\
&y_{rj}^l \le y_{rj} \le y_{rj}^u && \forall r, j && \text{(f)} \\
&\underline{w}_{kj}^l \le w_{kj} \le w_{kj}^u && \forall k, j \in J_{in}^k && \text{(g)} \\
&w_{kj}^l \le w_{kj} \le \overline{w}_{kj}^u && \forall k, j \in J_{out}^k && \text{(h)} \\
&\alpha_k, u_r, v_i \ge 0 && \forall r, i, k, j && \text{(i)}
\end{aligned} \tag{15}$$

where x_{ij}, y_{rj} and w_{kj} are decision variables associated with input, output and dual-role factors, and α_k represents the weight of k^{th} dual-role factor. The objective function of (15) maximizes the aggregate efficiency of all DMUs, which is a condensed form of $\sum_{k=1}^{K}\left[\sum_{j\in J_{out}^k}\left(\sum_{r=1}^{s} u_r y_{rj} + \alpha_k w_{kj}\right) + \sum_{j\in J_{out}^k}\left(\sum_{r=1}^{s} u_r y_{rj} - \alpha_k w_{kj}\right) + \sum_{j\notin (J_{out}^k \cup J_{in}^k)}\left(\sum_{r=1}^{s} u_r y_{rj}\right)\right]$. Constraint (15.b) and constraints (15.c) ensure that the efficiency scores of the individual DMUs do not exceed unity. Since $|.|$ represents the absolute function in constraints (15.d), $W_k^L \le \sum_{j=1}^{n}\left|\pi_{kj}\right| w_{kj} \le W_k^U$ can be rewritten as $W_k^L \le \sum_{j\in J_{in}^k}^{n} w_{kj} + \sum_{j\in J_{out}^k}^{n} w_{kj} \le W_k^U$, which guarantees that the total allocation of the k^{th} dual-role factor varies within the interval $[W_k^L, W_k^U]$. Constraints (15.e) and (15.f) specify the lower and upper bounds of inputs and outputs, respectively. Constraint (15.g) and (15.h) restrict the amount of the factor allocated to members of set J_{in}^k and J_{out}^k to be placed within the interval $[\underline{w}_{kj}^l, w_{kj}^u]$ and $[w_{kj}^l, \overline{w}_{kj}^u]$, respectively. It is assumed that the DM defines an apt value of $\underline{w}_{kj}^l$ and $\overline{w}_{kj}^u$.

Note that model (15) is a NLP problem due to $u_r y_{rj}, v_i x_{ij}$ and $\alpha_k w_{kj}$, however, the LP problem (16) can be obtained if we define the change of variables $\hat{y}_{rj} = u_r y_{rj}, \hat{x}_{ij} = v_i x_{ij}$ and $\widehat{w}_{kj} = \alpha_k w_{kj}$.

$$\begin{array}{ll}
\max \sum_{j=1}^{n}\left[\sum_{r=1}^{s} \hat{y}_{rj} + \sum_{k=1}^{K} \pi_{kj}\widehat{w}_{kj}\right] & \\
\text{s.t.} & \\
\sum_{j=1}^{n}\left[\sum_{i=1}^{m} \hat{x}_{ij}\right] = 1 & \\
\sum_{r=1}^{s} \hat{y}_{rj} + \sum_{k=1}^{K} \pi_{kj}\widehat{w}_{kj} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0 & \forall j \\
\alpha_k W_k^L \le \sum_{j=1}^{n}\left|\pi_{kj}\right|\widehat{w}_{kj} \le \alpha_k W_k^U & \forall k \\
v_i x_{ij}^l \le \hat{x}_{ij} \le v_i x_{ij}^u & \forall k, j \\
u_r y_{rj}^l \le \hat{y}_{rj} \le u_r y_{rj}^u & \forall r, j \\
\alpha_k \underline{w}_{kj}^l \le \widehat{w}_{kj} \le \alpha_k w_{kj}^u & \forall r, j \in J_{in}^k \\
\alpha_k w_{kj}^l \le \widehat{w}_{kj} \le \alpha_k \overline{w}_{kj}^u & \forall r, j \in J_{out}^k \\
\widehat{w}_{kj}, \hat{y}_{rj}, \hat{x}_{ij}, \alpha_k, u_r, v_i \ge 0 & \forall r, i, k, j
\end{array} \qquad (16)$$

It is easy to verify that from the optimal solution $\widehat{w}_{kj}^*, \hat{y}_{rj}^*, \hat{x}_{ij}^*, \alpha_k^*, u_r^*,$ and v_i^* of the LP problem (16), we can determine the optimal solution $w_{kj}^* = \frac{\widehat{w}_{kj}^*}{\alpha_k^*}, y_{rj}^* = \frac{\hat{y}_{rj}^*}{u_r^*}$, and $x_{ij}^* = \frac{\hat{x}_{ij}^*}{v_i^*}$ for the NLP problem (15).

Analogues to the model of Cook et al. (2006), the members that do not belong to $J_{out}^k \cup J_{in}^k$ are in an equilibrium position, that is, these members desire neither to acquire nor to lose the k^{th} resource ($k = 1, \dots, K$). This condition is fulfilled by letting $\pi_{kj} = 0$ for $j \notin \left(J_{out}^k \cup J_{in}^k\right)$ which is equivalent to the exclusion of w_{kj} from model (15). Undoubtedly, if $\mathrm{DMU}_{j \in J_{out}^k}$ is able to obtain an increase in the upper bound of w_{kj} without changing other DMU allocations, then the interval efficiency for DMU_j cannot deteriorate further. However, those other allocations can vary.

3.4 An identical role of a dual-role for all DMUs: A fuzzy decision-making method

Solving model (14) provides a bounded interval $\left[\bar{\xi}_j^l, \xi_j^u\right]$ for DMU_j with a unique status for each dual-role factor; however, in some situations, the DM is seeking an identical designation for dual-role factors for all DMUs. At present, we exploit the interval efficiencies calculated from models (8) and (10) along with the fuzzy non-dimensionalization approach proposed by Bellman and Zadeh (1970) with the intent to simultaneously evaluate all DMUs. In effect, the application of Bellman and Zadeh (1970)'s approach to the interval efficiency scores for all DMUs renders the final decision constructive in line with obtaining harmonious solutions through fuzzy membership function. Assume that e_o^u and $\bar{e}_o^l$ are the upper and lower bounds of efficiency derived from models (8) and (10). In order to make a decision about the performance of a certain DMU, the DM may get stuck in the lower and upper bounds of its efficiency as well as different role of dual-role factors. To combat this complicated situation and maximize a satisfaction degree of the DM, the goal can be expressed by dint of a fuzzy set $\tilde{e}_j$ with the following membership function:

$$\mu_{\tilde{e}_j}(e_j) = \begin{cases} 0 & e_j \leq \bar{e}_j^l \\ \dfrac{e_j - \bar{e}_j^l}{e_j^u - \bar{e}_j^l} & \bar{e}_j^l \leq e_j < e_j^u \\ 1 & e_j^u \leq e_j \end{cases} \tag{17}$$

Therefore, if $e_j^u \leq e_j$ the DM finds it definitely satisfactory, and if $e_j \leq \bar{e}_j^l$ the lowest degree of satisfaction for the DM occurs. Note that, for simplicity but without losing much generality, we look upon the goal $\tilde{e}_j$ as a linear function but other functions can be used without difficulty.

Based on the fuzzy max-min criterion of Bellman and Zadeh (1970), if $G_1, \dots, G_Q$ are Q different fuzzy goals on the space of alternatives X, with π_{G_q} as the membership function of $G_q, q = 1, \dots, Q$, then the associated *fuzzy decision* is defined as $D = \bigcap_{q=1}^{Q} G_q$ with the membership function $\pi_D(x) = \min_{q=1,\dots,Q}\left\{\pi_{G_q}(x)\right\}$. For a fuzzy decision D on X, an alternative $x^* \in X$ is optimal, if $\pi_D(x^*) = \max_{x\in X}\{\pi_D(x)\} = \max_{x\in X}\left\{\min_{q=1,\dots,Q}\left\{\pi_{G_q}(x)\right\}\right\}$.

Presently, we apply the above Bellman-Zadeh's methodology on fuzzy goals $\tilde{e}_j, (j = 1, \dots, n)$, in which the space of alternatives for the goals is the set of all input, output and dual-role variables (i.e., x_{ij}, y_{rj} and w_{kj}) and their associated weights (i.e. $v_i, u_r, \delta_k, \gamma_k$). Thereby, we arrive at the following integrated DEA-based model to simultaneously evaluate all DMUs in the presence of dual-role factors:

$$\begin{aligned}
&\max\left\{\min_{j=1,\ldots,n}\left\{\mu_{\tilde{e}_j}\left(\frac{\sum_{r=1}^{s}u_r y_{rj}+\sum_{k=1}^{K}\gamma_k w_{kj}-\sum_{k=1}^{K}\delta_k w_{kj}}{\sum_{i=1}^{m}v_i x_{ij}}\right)\right\}\right\}\\
&\text{s.t.}\\
&\textstyle\sum_{r=1}^{s}u_r y_{rj}+\sum_{k=1}^{K}\gamma_k w_{kj}-\sum_{k=1}^{K}\delta_k w_{kj}-\sum_{i=1}^{m}v_i x_{ij}\le 0 && \forall j\\
&\textstyle\sum_{i=1}^{m}v_i x_{ij}\le 1 && \forall j\\
&y_{rj}^{l}\le y_{rj}\le y_{rj}^{u} && \forall r,j\\
&w_{kj}^{l}\le w_{kj}\le w_{kj}^{u} && \forall k,j\\
&x_{ij}^{l}\le x_{ij}\le x_{ij}^{u} && \forall i,j\\
&u_r, v_i, \gamma_k, \delta_k\ge 0 && \forall r,i,k
\end{aligned}\tag{18}$$

This model aims to find a feasible solution which maximizes the minimum membership degrees of efficiency for all DMUs. Model (18) is a non-linear fractional programming problem due to $u_r y_{rj}, v_i x_{ij}, \gamma_k w_{kj}$ and $\delta_k w_{kj}$, and the fractions form of the objective function. We simplify model (18) by incorporating the following variable substitutions:

$$\begin{aligned}
x_{ij} &= x_{ij}^{l}+f_{ij}\left(x_{ij}^{u}-x_{ij}^{l}\right) && \forall i,j\\
y_{rj} &= y_{rj}^{l}+g_{rj}\left(y_{rj}^{u}-y_{rj}^{l}\right) && \forall r,j\\
w_{kj} &= w_{kj}^{l}+h_{kj}\left(w_{kj}^{u}-w_{kj}^{l}\right) && \forall k,j
\end{aligned}$$

$$\eta=\min\left\{\mu_{\tilde{e}_j}\left(\frac{\sum_{r=1}^{s}u_r y_{rj}+\sum_{k=1}^{K}\gamma_k w_{kj}-\sum_{k=1}^{K}\delta_k w_{kj}}{\sum_{i=1}^{m}v_i x_{ij}}\right): j=1,\ldots,n\right\}$$

where $0\le f_{ij}\le 1$, $0\le g_{rj}\le 1$ and $0\le h_{kj}\le 1$. Taking the last variable substitutions into consideration, we have

$$\eta\le\mu_{\tilde{e}_j}\left(\frac{\sum_{r=1}^{s}u_r y_{rj}+\sum_{k=1}^{K}\gamma_k w_{kj}-\sum_{k=1}^{K}\delta_k w_{kj}}{\sum_{i=1}^{m}v_i x_{ij}}\right),\quad \forall j$$

or

$$\eta\le\frac{\boldsymbol{u}\hat{\boldsymbol{y}}_j-\boldsymbol{\gamma}\hat{\boldsymbol{w}}_j+\boldsymbol{\delta}\hat{\boldsymbol{w}}_j-\bar{e}_j^{l}\left(\boldsymbol{v}\hat{\boldsymbol{x}}_j\right)}{\left(e_j^{u}-\bar{e}_j^{l}\right)\left(\boldsymbol{v}\hat{\boldsymbol{x}}_j\right)},\qquad \forall j$$

where $\boldsymbol{u}=(u_1,\ldots,u_s)$, $\hat{\boldsymbol{y}}_j=\left(y_{1j}^{l}+g_{1j}\left(y_{1j}^{u}-y_{1j}^{l}\right),\ldots,y_{sj}^{l}+g_{sj}\left(y_{sj}^{u}-y_{sj}^{l}\right)\right)$, $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_K)$, $\hat{\boldsymbol{w}}_j=\left(w_{1j}^{l}+h_{1j}\left(w_{1j}^{u}-w_{1j}^{l}\right),\ldots,w_{Kj}^{l}+h_{Kj}\left(w_{Kj}^{u}-w_{Kj}^{l}\right)\right)$, $\boldsymbol{v}=(v_1,\ldots,v_m)$, and $\hat{\boldsymbol{x}}_j=\left(x_{1j}^{l}+f_{1j}\left(x_{1j}^{u}-x_{1j}^{l}\right),\ldots,x_{mj}^{l}+f_{mj}\left(x_{mj}^{u}-x_{mj}^{l}\right)\right)$.

By these substitutions, (18) becomes the following NLP problem:

$$
\begin{aligned}
&\max \quad \eta \\
&\text{s.t.} \\
&\eta\left(e_j^u-\bar{e}_j^l\right)\textstyle\sum_{i=1}^m v_i\left[x_{ij}^l+f_{ij}\left(x_{ij}^u-x_{ij}^l\right)\right] \le \sum_{r=1}^s u_r\left[y_{rj}^l+g_{rj}\left(y_{rj}^u-y_{rj}^l\right)\right]- \\
&\textstyle\sum_{k=1}^K \gamma_k\left[w_{kj}^l+h_{kj}\left(w_{kj}^u-w_{kj}^l\right)\right]+\sum_{k=1}^K \delta_k\left[w_{kj}^l+h_{kj}\left(w_{kj}^u-w_{kj}^l\right)\right]- \\
&\qquad \bar{e}_j^l \times \textstyle\sum_{i=1}^m v_i\left[x_{ij}^l+f_{ij}\left(x_{ij}^u-x_{ij}^l\right)\right] && \forall j \\
&\textstyle\sum_{r=1}^s u_r\left[y_{rj}^l+g_{rj}\left(y_{rj}^u-y_{rj}^l\right)\right]+\sum_{k=1}^K \gamma_k\left[w_{kj}^l+h_{kj}\left(w_{kj}^u-w_{kj}^l\right)\right] && (19) \\
&\qquad \textstyle\sum_{k=1}^K \delta_k\left[w_{kj}^l+h_{kj}\left(w_{kj}^u-w_{kj}^l\right)\right]-\sum_{i=1}^m v_i\left[x_{ij}^l+f_{ij}\left(x_{ij}^u-x_{ij}^l\right)\right] \le 0 && \forall j \\
&\textstyle\sum_{i=1}^m v_i\left[x_{ij}^l+f_{ij}\left(x_{ij}^u-x_{ij}^l\right)\right] \le 1 && \forall j \\
&0 \le \eta \le 1 \\
&u_r, v_i, \gamma_k, \delta_k \ge 0 && \forall r,i,k \\
&0 \le f_{ij}, g_{rj}, h_{kj} \le 1 && \forall r,i,k,j
\end{aligned}
$$

Since model (19) is still non-linear, let us make the following variable substitutions

$$
\begin{aligned}
p_{rj}^1 &= u_r g_{rj} && \forall r,j \\
p_{kj}^2 &= \gamma_k h_{kj} && \forall k,j \\
p_{kj}^3 &= \delta_k h_{kj} && \forall k,j \\
p_{ij}^4 &= v_i f_{ij} && \forall i,j
\end{aligned}
$$

where $0 \le p_{rj}^1 \le u_r, 0 \le p_{kj}^2 \le \gamma_k$, $0 \le p_{kj}^3 \le \delta_k$ and $0 \le p_{ij}^4 \le v_i$, and applying them to model (19) leads to the further simplified model (20):

$$
\begin{aligned}
&\max \quad \eta \\
&\text{s.t.} \\
&\eta\left(e_j^u-\bar{e}_j^l\right)\textstyle\sum_{i=1}^m\left[v_i x_{ij}^l+p_{ij}^4\left(x_{ij}^u-x_{ij}^l\right)\right] \le \sum_{r=1}^s\left[u_r y_{rj}^l+p_{rj}^1\left(y_{rj}^u-y_{rj}^l\right)\right]+ \\
&\textstyle\sum_{k=1}^K\left[\gamma_k w_{kj}^l+p_{kj}^2(w_{kj}^u-w_{kj}^l)\right]-\sum_{k=1}^K\left[\delta_k w_{kj}^l+p_{kj}^3(w_{kj}^u-w_{kj}^l)\right]- \\
&\qquad \bar{e}_j^l \times \textstyle\sum_{i=1}^m\left[v_i x_{ij}^l+p_{ij}^4\left(x_{ij}^u-x_{ij}^l\right)\right] && \forall j \\
&\textstyle\sum_{r=1}^s\left[u_r y_{rj}^l+p_{rj}^1\left(y_{rj}^u-y_{rj}^l\right)\right]+\sum_{k=1}^K\left[\gamma_k w_{kj}^l+p_{kj}^2(w_{kj}^u-w_{kj}^l)\right]- \\
&\qquad \textstyle\sum_{k=1}^k\left[\delta_k w_{kj}^l+p_{kj}^3(w_{kj}^u-w_{kj}^l)\right]-\sum_{i=1}^m\left[v_i x_{ij}^l+p_{ij}^4\left(x_{ij}^u-x_{ij}^l\right)\right] \le 0 && \forall j \\
&\textstyle\sum_{i=1}^m\left[v_i x_{ij}^l+p_{ij}^4\left(x_{ij}^u-x_{ij}^l\right)\right] \le 1 && \forall j \qquad (20) \\
&0 \le p_{rj}^1 \le u_r && \forall r,j \\
&0 \le p_{kj}^2 \le \gamma_k && \forall k,j \\
&0 \le p_{kj}^3 \le \delta_k && \forall k,j \\
&0 \le p_{ij}^4 \le v_i && \forall i,j \\
&0 \le \eta \le 1 \\
&u_r, v_i \ge 0 && \forall r,i \\
&\gamma_k, \delta_k \ge 0 && \forall k
\end{aligned}
$$

Analogous to model (6), γ_k and δ_k might simultaneously be non-zero in the optimal solution because these variables are independent. The same idea for dealing with model (6) can be employed by introducing auxiliary binary variables and model (20) is eventually reformulated to the following non-linear problem:

$$
\begin{aligned}
&\max \quad \eta \\
&\text{s.t.} \\
&\eta\left(e_j^u - \bar{e}_j^l\right) \textstyle\sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \leq \textstyle\sum_{r=1}^{s}\left[u_r y_{rj}^l + p_{rj}^1\left(y_{rj}^u - y_{rj}^l\right)\right] + \\
&\textstyle\sum_{k=1}^{K}\left[b_k\gamma_k w_{kj}^l + p_{kj}^2(w_{kj}^u - w_{kj}^l)\right] - \textstyle\sum_{k=1}^{K}\left[d_k\delta_k w_{kj}^l + p_{kj}^3\left(w_{kj}^u - w_{kj}^l\right)\right] - \\
&\qquad \bar{e}_j^l \times \textstyle\sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] && \forall j \\
&\textstyle\sum_{r=1}^{s}\left[u_r y_{rj}^l + p_{rj}^1\left(y_{rj}^u - y_{rj}^l\right)\right] + \textstyle\sum_{k=1}^{K}\left[b_k\gamma_k w_{kj}^l + p_{kj}^2(w_{kj}^u - w_{kj}^l)\right] - \\
&\qquad \textstyle\sum_{k=1}^{K}\left[d_k\delta_k w_{kj}^l + p_{kj}^3(w_{kj}^u - w_{kj}^l)\right] - \textstyle\sum_{i=1}^{m}\left[v_i x_{lj}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \leq 0 && \forall j \\
&\textstyle\sum_{i=1}^{m}\left[v_i x_{lj}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \leq 1 && \forall j \\
&0 \leq p_{rj}^1 \leq u_r && \forall r,j \\
&0 \leq p_{kj}^2 \leq \gamma_k && \forall k,j \\
&0 \leq p_{kj}^3 \leq \delta_k && \forall k,j \\
&0 \leq p_{ij}^4 \leq v_i && \forall i,j \\
&0 \leq \eta \leq 1 \\
&b_k + d_k \leq 1 && \forall k \\
&b_k, d_k \in \{0,1\} && \forall k \\
&\gamma_k, \delta_k \geq 0 && \forall k \\
&u_r, v_i, \geq 0 && \forall r,i
\end{aligned}
\tag{21}
$$

Although the terms $\eta\left(e_j^u - \bar{e}_j^l\right) \sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right]$, $b_k\gamma_k$ and $d_k\delta_k$ make the above model non-linear, we can use a linearization technique to eliminate the last two non-linearity terms. To what follows, letting $\hat{\gamma}_k = b_k\gamma_k$ and $\hat{\delta}_k = d_k\delta_k$, model (21) is rewritten as follows:

$$
\begin{array}{ll}
\max \quad \eta & \\
\text{s.t.} & \\
\eta\left(e_j^u - \bar{e}_j^l\right) \sum_{i=1}^m \left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \leq \sum_{r=1}^s \left[u_r y_{rj}^l + p_{rj}^1\left(y_{rj}^u - y_{rj}^l\right)\right] + & \\
\quad \sum_{k=1}^K \left[\hat{\gamma}_k w_{kj}^l + p_{kj}^2 (w_{kj}^u - w_{kj}^l)\right] - \sum_{k=1}^K \left[\hat{\delta}_k w_{kj}^l + p_{kj}^3 (w_{kj}^u - w_{kj}^l)\right] - & \\
\quad \bar{e}_j^l \times \sum_{i=1}^m \left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] & \forall j \\
\sum_{r=1}^s \left[u_r y_{rj}^l + p_{rj}^1\left(y_{rj}^u - y_{rj}^l\right)\right] + \sum_{k=1}^K \left[\hat{\gamma}_k w_{kj}^l + p_{kj}^2 (w_{kj}^u - w_{kj}^l)\right] - & \\
\quad \sum_{k=1}^K \left[\hat{\delta}_k w_{kj}^l + p_{kj}^3 (w_{kj}^u - w_{kj}^l)\right] - \sum_{i=1}^m \left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \leq 0 & \forall j \\
\sum_{i=1}^m \left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \leq 1 & \forall j \\
0 \leq p_{rj}^1 \leq u_r & \forall r, j \\
0 \leq p_{kj}^2 \leq \hat{\gamma}_k & \forall k, j \\
0 \leq p_{kj}^3 \leq \hat{\delta}_k & \forall k, j \\
0 \leq p_{ij}^4 \leq v_i & \forall i, j \\
0 \leq \eta \leq 1 & \\
0 \leq \hat{\gamma}_k \leq M b_k & \forall k \\
0 \leq \hat{\delta}_l \leq M d_k & \forall k \\
b_k + d_k \leq 1 & \forall k \\
b_k, d_k \in \{0,1\} & \forall k \\
u_r, v_i \geq 0 & \forall r, i
\end{array}
\tag{22}
$$

For the sake of more clarity of variable transformations employed in models (19) to (22), it should be noted that although the variables h_{kj}, γ_k and δ_k are used in the variable transformations twice, this does not lead to the inconsistent solutions calculated due to the special structures of these models.

In model (22), the binary variable d_k and b_k relate to the ways for showing that w_k is functioning as an input and output, respectively, in which $b_k = 1$ and $d_k = 1$ show that w_k as an input and output, respectively. Accordingly, if $b_k = 0$, then $\hat{\gamma}_k = p_{kj}^2 = 0$ which implies that $\sum_{k=1}^K [\hat{\gamma}_k w_{kj}^l + p_{kj}^2 (w_{kj}^u - w_{kj}^l)] = 0$ and hence the dual-role factor w_k cannot serve as an output. Besides, in case of $d_k = 0$, the amount of $\hat{\delta}_k$ and p_{kj}^3 equals to zero which showcases that $\sum_{k=1}^K \left[\hat{\delta}_k w_{kj}^l + p_{kj}^3 (w_{kj}^u - w_{kj}^l)\right] = 0$ and w_k cannot play the role of an input. It should be noted that the model ignores w_k serving as the input and output simultaneously because $b_k = d_k = 1$ is impossible to occur in view of the constraint $b_k + d_k \leq 1$.

Model (22) is still non-linear due to $\eta\left(e_j^u - \bar{e}_j^l\right) \sum_{i=1}^m \left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right]$ $(j = 1, \ldots, n)$ and it is easy to verify that its trivial optimal solution is

$(\eta^*, \boldsymbol{v}^*, \boldsymbol{u}^*, \widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\delta}}^*, \boldsymbol{p^1}^*, \boldsymbol{p^2}^*, \boldsymbol{p^3}^*, \boldsymbol{p^4}^*, \boldsymbol{b}^*, \boldsymbol{d}^*) = (1, \mathbf{0}_{m(n+1)+s(n+1)+K(6+2n)})$[1]. One way to obtain a non-trivial solution is to impose a positive lower bound on weights. However, finding a suitable value for the lower bound is intricate since infeasibility may occur in case of an improper lower bound. We therefore propose an intuitive approach to gain a non-trivial optimal solution for model (22) without the need of considering the lower bound.

We develop the MILP model (23) for finding a non-trivial solution for model (22). Contrary to (22), model (23) maximizes $\sum_{j=1}^{n}\sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right]$ rather than η for the purpose of calculating the non-zero vector of input and output weights.

$$
\begin{array}{ll}
\max\ z = \sum_{j=1}^{n}\sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] & \\
\text{s.t.} & \\
\bar{\eta}\left(e_j^u - \bar{e}_j^l\right)\sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \le \sum_{r=1}^{s}\left[u_r y_{rj}^l + p_{rj}^1\left(y_{rj}^u - y_{rj}^l\right)\right] + & \\
\sum_{k=1}^{K}\left[\hat{\gamma}_k w_{kj}^l + p_{kj}^2(w_{kj}^u - w_{kj}^l)\right] - \sum_{k=1}^{K}\left[\hat{\delta}_k w_{kj}^l + p_{kj}^3(w_{kj}^u - w_{kj}^l)\right] - & \\
\bar{e}_j^l \times \sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] & \forall j \\
\sum_{r=1}^{s}\left[u_r y_{rj}^l + p_{rj}^1\left(y_{rj}^u - y_{rj}^l\right)\right] + \sum_{k=1}^{K}\left[\hat{\gamma}_k w_{kj}^l + p_{kj}^2(w_{kj}^u - w_{kj}^l)\right] - & \\
\sum_{k=1}^{K}\left[\hat{\delta}_k w_{kj}^l + p_{kj}^3(w_{kj}^u - w_{kj}^l)\right] - \sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \le 0 & \forall j \\
\sum_{i=1}^{m}\left[v_i x_{ij}^l + p_{ij}^4\left(x_{ij}^u - x_{ij}^l\right)\right] \le 1 & \forall j \\
0 \le p_{rj}^1 \le u_r & \forall r, j \\
0 \le p_{kj}^2 \le \hat{\gamma}_k & \forall k, j \\
0 \le p_{kj}^3 \le \hat{\delta}_k & \forall k, j \\
0 \le p_{ij}^4 \le v_i & \forall i, j \\
0 \le \hat{\gamma}_k \le M b_k & \forall k \\
0 \le \hat{\delta}_k \le M d_k & \forall k \\
b_k + d_k \le 1 & \forall k \\
b_k, d_k \in \{0,1\} & \forall k \\
u_r, v_i \ge 0 & \forall r, i
\end{array}
\tag{23}
$$

where $\bar{\eta}$ is a parameter that varies within [0, 1]. Suppose that the optimal objective value of model (23) is positive (i.e., $z^* > 0$) and $\left(\overline{\boldsymbol{v}}, \overline{\boldsymbol{u}}, \overline{\widehat{\boldsymbol{\gamma}}}, \overline{\widehat{\boldsymbol{\delta}}}\right)$ is the associated vector of optimal weights[2]. Clearly, $\left(\bar{\eta}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{u}}, \overline{\widehat{\boldsymbol{\gamma}}}, \overline{\widehat{\boldsymbol{\delta}}}\right)$ is a non-trivial feasible solution of model (22).

---

[1] $\mathbf{0}_p$ is the origin in $\mathbb{R}^p$ space, i.e. $\mathbf{0}_p = (0, \ldots, 0) \in \mathbb{R}^p$.

[2] For the sake of simplicity and brevity, we utilize $(\eta, \boldsymbol{v}, \boldsymbol{u}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\delta}})$ for presenting the feasible solution rather than $(\eta, \boldsymbol{v}, \boldsymbol{u}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\delta}}, \boldsymbol{p^1}, \boldsymbol{p^2}, \boldsymbol{p^3}, \boldsymbol{p^4}, \boldsymbol{b}, \boldsymbol{d})$ where $\boldsymbol{p^1}, \boldsymbol{p^2}, \boldsymbol{p^3}, \boldsymbol{p^4}, \boldsymbol{b}$ and $\boldsymbol{d}$ are auxiliary variables.

We for the present require to seek out a maximal value of $\bar{\eta}$ in model (23) to find the optimal non-trivial solution for model (22). To do so, the following iterative bisection algorithm is presented to determine a maximal value for $\bar{\eta}$:

Proposed Algorithm:

Input:

- *Lower and upper bounds of data*: $\{\boldsymbol{x}_{ij}^{l}, \boldsymbol{x}_{ij}^{u}, \boldsymbol{y}_{rj}^{l}, \boldsymbol{y}_{rj}^{u}, \boldsymbol{w}_{kj}^{l}, \boldsymbol{w}_{kj}^{u}, \bar{\boldsymbol{e}}_{j}^{l}, \boldsymbol{e}_{j}^{u}: \boldsymbol{i} = \boldsymbol{1}, \ldots, \boldsymbol{m};\ \boldsymbol{r} = \boldsymbol{1}, \ldots, \boldsymbol{s};\ \boldsymbol{k} = \boldsymbol{1}, \ldots, \boldsymbol{K};\ \boldsymbol{j} = \boldsymbol{1}, \ldots, \boldsymbol{n}\}$.
- *Stopping criterion*: $|\boldsymbol{\eta}^{(t)} - \boldsymbol{\eta}^{(t-1)}| \leq \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} > \boldsymbol{0}$ is a sufficiently small constant.

Step 0. Let $\eta_{min} = 0, \eta_{max} = 2, t = -1$.

Step 1. Let $t = t + 1, \eta^{(t)} = (\eta_{min} + \eta_{max})/2$.

Step 2. Solve model (23) with $\bar{\eta} = \eta^{(t)}$ and find the optimal solution ($\boldsymbol{v}^{(t)}, \boldsymbol{u}^{(t)}, \hat{\boldsymbol{\gamma}}^{(t)}, \hat{\boldsymbol{\delta}}^{(t)}$). If the optimal solution is trivial (zero), let $\eta_{max} = \eta^{(t)}$ and go to Step 1.

Step 3. If the stopping criterion is not satisfied, let $\eta_{min} = \eta^{(t)}$ and go to Step 1.

Output. $(\eta^{(t)}, \boldsymbol{v}^{(t)}, \boldsymbol{u}^{(t)}, \hat{\boldsymbol{\gamma}}^{(t)}, \hat{\boldsymbol{\delta}}^{(t)})$ as an optimal solution of model (22).

We base the above algorithm on a bisection approach in order to search the interval [0,1] for the purpose of finding maximum value of $\bar{\eta}$ such that model (23) has a non-trivial optimal solution. Given that the superscript t denotes the iteration number, $(\eta^{(t)}, \boldsymbol{v}^{(t)}, \boldsymbol{u}^{(t)}, \hat{\boldsymbol{\gamma}}^{(t)}, \hat{\boldsymbol{\delta}}^{(t)})$ shows the t^{th} approximation of the optimal solution. The aim of the algorithm is to operate iteratively to generate a sequence of feasible solutions $(\eta^{(t)}, \boldsymbol{v}^{(t)}, \boldsymbol{u}^{(t)}, \hat{\boldsymbol{\gamma}}^{(t)}, \hat{\boldsymbol{\delta}}^{(t)})$ which finally converges to an optimal solution of model (22).

Consider $\bar{\eta} = \eta^{(t)}$ as a fixed value. If the optimal solution of model (23) is trivial, then Step 2 decreases the value of $\bar{\eta}$ and the current search interval is changed from $[\eta_{min}, \eta_{max}]$ to $\left[\eta_{min}, \frac{\eta_{min}+\eta_{max}}{2}\right]$, otherwise, $\bar{\eta}$ can be considered as a lower bound for the optimal objective value of model (22). In order to verify whether the value of $\bar{\eta}$ can be increased, Step 3 bisects the current search interval $[\eta_{min}, \eta_{max}]$ to yield $\left[\frac{\eta_{min}+\eta_{max}}{2}, \eta_{max}\right]$. In other word, the search interval $[\eta_{min}, \eta_{max}]$ that contains the optimal value of η is halved in each iteration. Thereby, the stopping criterion $\left|\eta^{(t)} - \eta^{(t-1)}\right| \le \varepsilon$ will be finally satisfied and $\{\eta^{(t)}\}$ will be converged to the optimal objective value of model (22). The schematic of the proposed algorithm is depicted in Figure 1.

---Insert Figure 1 Here---

The proposed algorithm seeks a suitable value for parameter $\bar{\eta}$ in MILP model (23) which can be consider as an appropriate approximation for the optimal objective value of model (22). It should be noted that the optimal objective value of model (22) is $\bar{\eta} = \max\left\{\min\left\{\mu_{\tilde{e}_j}(e_j)\right\}, j = 1, \dots, n\right\}$ which leads to the optimal common set of weights in a way that the worst membership degree has its maximum (best) value. This property is known as a '*best-of-the-worst*' rule in *maximin* problems used in decision theory. Therefore, the algorithm results in the optimal common set of weights $(\boldsymbol{v}^*, \boldsymbol{u}^*, \widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\delta}}^*)$ which enables us to measure the efficiency score of j^{th} DMU as $e_j = \frac{\boldsymbol{u}^*\boldsymbol{y}_j + \widehat{\boldsymbol{\gamma}}^*\boldsymbol{w}_j - \widehat{\boldsymbol{\delta}}^*\boldsymbol{w}_j}{\boldsymbol{v}^*\boldsymbol{x}_j}$ as well as to secure its membership degree of using equation (17). Given that model (22) is an integrated model, the role of a dual-role factor w_k would be unique for all the DMUs by means of the optimal values of $\hat{\gamma}_k^*$ and $\hat{\delta}_k^*$, that is, if $\hat{\gamma}_k^* > 0$ then w_k is treated as an output and if $\hat{\delta}_k^* > 0$ then w_k plays an input role; otherwise (i.e. $\hat{\gamma}_k^* = \hat{\delta}_k^* = 0$) w_k is in the equilibrium situation.

4 Application

In this section, we evaluate 20 banks in Gulf Cooperation Council countries (GCC) to illustrate the merits of the proposed approach in our study. The data set taken from Hatami-Marbini et al. (2014) consists of three interval inputs (*total assets*, *Equity* and *deposits*) and two interval outputs (*loans* and *net profit*). However, we disregard *net profit* factor which includes the negative values along with considering *deposit* as a dual-role factor. Table 1 shows the interval dataset of 20 banks[3].

[3] All models have been solved using CPLEX solver of GAMS software (See Appendix).

---Insert Table 1 Here---

To emphasize the importance of uncertainty in the proposed framework, in comparison with the deterministic approaches in the literature, we analyse five scenarios by considering various data settings. The first four scenarios concern the situation where the crisp data are used to measure the efficiency of 20 banks in the way that these scenarios take account of the lower and upper bounds of the interval data. The last scenario focuses on the proposed method in this research where the observed input–output data and dual-role factor are imprecise.

4.1 Precise Scenarios

In the first two precise scenarios, we assume that the inputs and outputs of 20 banks take their lower bounds x_{ij}^{l} and upper bounds y_{rj}^{u}, respectively. The difference of the first and second scenarios stems from defining the dual-role factor (deposit) which takes their lower bounds w_{kj}^{l} for the first scenario and the upper bounds w_{kj}^{u} for the second scenario. The last two precise scenarios are oppositely presumed by taking the lower bounds y_{rj}^{l} for the outputs and the upper bounds for the inputs x_{ij}^{u} into account in which the third and fourth scenarios include the upper bounds w_{kj}^{u} and lower bound w_{kj}^{l} of the dual-role factor, respectively. As a result, the four precise scenarios represent the deterministic efficient frontiers by means of $\{x_{ij}^{l}, y_{rj}^{u}, w_{kj}^{l}\}$, $\{x_{ij}^{l}, y_{rj}^{u}, w_{kj}^{u}\}$, $\{x_{ij}^{u}, y_{rj}^{l}, w_{kj}^{u}\}$, and $\{x_{ij}^{u}, y_{rj}^{l}, w_{kj}^{l}\}$, respectively. Table 2 reports the precise efficiency scores of four precise scenarios by solving the deterministic DEA model (3) proposed by Cook et al. (2006).

---Insert Table 2 Here---

According to the results, apart from Banks 1, 3, 9 and 20 which are efficient for all the scenarios, Banks 10 and 17 in Scenario 1, Bank 19 in Scenario 2, and Banks 5 and 12 in Scenario 3 are efficient, while Bank 6 in Scenarios 1 and 3, and Bank 8 in Scenarios 2 and 4 have the worst performance among the banks. In addition, the ranking of the banks based on the efficiency score of each scenario are shown in Table 2. To determine the actual status of the dual-role factor w_{DEPO}, we need to focus on the optimal value of δ^*_{DEPO} and γ^*_{DEPO} after running model (3) as presented in Table 3. In all scenarios, the dual-role factor (deposit) is treated as a non-discretionary input for Banks 1, 3 and 20 since their value for δ^*_{DEPO} is positive while it is functioned as an output for other banks due to $\gamma^*_k > 0$.

---Insert Table 3 Here---

4.2 Imprecise Scenario

Given the imprecise data in this scenario, we first solve models (8) and (9) to calculate the upper and lower bounds of efficiencies of the banks as reported in the "e^u_j" and "e^l_j" columns of Table 4. The status of *deposit* (dual-role factor) concerns the optimal value of b_{DEPO} and d_{DEPO} variables. The "b^*_{DEPO}" and "d^*_{DEPO}" columns in Table 4 showcase their optimal values for all the banks, in which the deposit is treated as an output and input if $\{b^*_{DEPO} = 1, d^*_{DEPO} = 0\}$ and $\{b^*_{DEPO} = 0, d^*_{DEPO} = 1\}$, respectively, for a given bank. Note that the *deposit* factor is at the equilibrium status if $\{b^*_{DEPO} = d^*_{DEPO} = 0\}$.

---Insert Table 4 Here---

Apart from Bank 8, the value of e^u_j for all the banks is equal to one. Therefore, the upper bounds of efficiencies in this situation cannot act as a focal point for evaluating the units. In such case, the lower efficiency e^l_j is of interest to us, although the values reported in the "e^l_j" column of Table 4 may not be the lowest efficiency as showed in Theorem 1. We hence re-calculate the lower bounds of efficiencies for the banks using the proposed model (10) as presented in the " $\bar{e}^l_j$" column of Table 4 . As can be observed from the results, the lower efficiency obtained from model (10) is always less than or equal to the efficiency derived from model (9) (c.f., Theorem 1). In addition, model (10) improves the discriminatory power contrary to the results calculated from model (9). After running model (10), we enable to determine the status of the *deposit* by dint of the optimal value of b_{DEPO}^{*} and d_{DEPO}^{*} in which it is acting as an input and output if $\{b_{DEPO}^{*}=0, d_{DEPO}^{*}=1\}$ and $\{b_{DEPO}^{*}=1, d_{DEPO}^{*}=0\}$, respectively, but *deposit* is at the equilibrium status in case of $b_{DEPO}^{*}=d_{DEPO}^{*}=0$. As a result, from the pessimistic standpoint (lower efficiency) and the results reported in the 3rd and 4th columns of Table 4, the deposit factor is viewed as an input for Banks 3, 7, 15 and 17 while it is at the equilibrium status for the outstanding banks. From the optimistic viewpoint (upper efficiency) and the results reported in the 9th and 10th columns of Table 4, the deposit factor is designated as an input for Bank 1, is at the equilibrium status for Banks 3, 7, 14 and 20 and designated as an output for the outstanding banks.

As can be seen, the role of the deposit with respect to the values of $\bar{e}_j^l, e_j^l$, and e_j^u is not identical for the majority of DMUs. To deal with the problem, we solve model (14) and find $\bar{\xi}_o^l=\theta^*$ and $\xi_o^u=\sum_{r=1}^{s}u_r^* y_{ro}^u+\sum_{k=1}^{K}\hat{\gamma}_k^* w_{ko}^u-\sum_{k=1}^{K}\hat{\delta}_k^* w_{ko}^l$ along with $b_{DEPO}^{*}, d_{DEPO}^{*}$ as shown in the last four columns of Table 4. The interval efficiencies obtained from model (14), i.e. $[\bar{\xi}_j^l, \xi_j^u]$, are a subset of the interval efficiencies obtained from models (8) and (10), i.e., $[\bar{e}_j^l, e_j^u]$. Given the results derived from model (14), the deposit considered as an output for Banks 4, 6, 7, 8, 10, 13, 15, 16, 17 and 20 and is at the equilibrium status for the other banks.

4.3 Reallocation of deposit

Taking into consideration the results computed from model (14), the following sets can be defined: $J_{out}^{DEPO}=\{\text{Bank 4, Bank 6, Bank 7, Bank 8, Bank 10, Bank 13, Bank 15, Bank 16, Bank 17, Bank 20}\}$ and $J_{in}^{DEPO}=\emptyset$. We assume a 10% increase in $w_{DEPO,j}^u$ for $j\in J_{out}^{DEPO}$ to obtain $[w_{DEPO,j}^l, \bar{w}_{DEPO,j}^u]$ which indicates the total possible bounded resource, i.e., $[W_{DEPO}^L, W_{DEPO}^U]=\left[\sum_{j\in J_{out}^{DEPO}} w_{DEPO,j}^l, 1.1\times\left(\sum_{j\in J_{out}^{DEPO}} w_{DEPO,j}^u\right)\right]=[48907.90, 57786.85]$. Table 5 shows the result after applying model (15).

---Insert Table 5 Here---

The second and fourth columns of Table 5 show the current allocation and new reallocation of deposit in the form of interval values, respectively, and the third and fifth columns of Table 5 report the current and new interval efficiencies. The changed values are in bold. As can be seen, there is no efficiency shift in the lower bound of reallocation even though the upper bounds of reallocation for several banks are changed. We therefore calculate the ratio of the new efficiency to current efficiency, as presented in "ratio" column of Table 5, to determine the efficiency change through the reallocation process. Whilst in the majority of cases the resulting reallocation remains unchanged, the efficiency of 30% of banks are augmented (in bold in Table 5) due to the reallocation exercise and deposit changes in 50% of banks (listed in bold in the last column of Table 5).

4.4 An identical role of deposit for all banks

As such, the ranking order of the banks can be provided based on the lower bounds of the efficiencies $\bar{e}_j^l$ although we are in need of taking Bellman-Zadeh's approach into account due to the dearth of compatibility of identifying the status of the dual-role factor using models (8) and (10). In this regard, we use the interval efficiency to define the fuzzy goal for each back. For instance, the interval efficiency of Bank 1 calculated from models (8) and (10) is $[\bar{e}_1^l, e_1^u] = [0.68414,1]$ and its fuzzy goal is expressed as follows:

$$\mu_{\tilde{e}_1}(e_1) = \begin{cases} 0 & e_1 \leq 0.68414 \\ \dfrac{e_1 - 0.68414}{1 - 0.68414} & 0.68414 \leq e_1 \leq 1 \\ 1 & 1 \leq e_1 \end{cases}$$

where $e_1 = (u_{\mathrm{LOAN}}^* y_{\mathrm{LOAN},1}^* + \gamma_{\mathrm{DEPO}}^* w_{\mathrm{DEPO},1}^* - \delta_{\mathrm{DEPO}}^* w_{\mathrm{DEPO},1}^*)/(v_{\mathrm{ASST}}^* x_{\mathrm{ASST},1}^* + v_{\mathrm{EQTY}}^* x_{\mathrm{EQTY},1}^*)$. The membership function $\mu_{\tilde{e}_1}(e_1)$ is depicted in Figure 2.

---Insert Figure 2 Here---

We hence need to solve model (18) to acquire the optimal common weights (i.e. $v_i^*, u_r^*, \delta_k^*, \gamma_k^*$) and identical role of deposit. In doing so, we apply the proposed algorithm to find the optimal value of η and its associated weights in model (22). To implement the algorithm, we set the "stopping criterion" as $|\eta^{(t)} - \eta^{(t-1)}| \leq 10^{-4}$ as well as taking into account interval values of inputs, outputs and dual-roles factors (see Table 1) and the interval efficiencies $[\bar{e}_j^l, e_j^u]$ ($j = 1, \dots, 20$) obtained from models (8) and (10) (see Table 4). The detailed execution of the algorithm with $t = 14$ is reported in Table 6. For instance, the first three iterations for moving toward the maximal value of $\bar{\eta}$ in model (23) are described below:

First iteration:

Step 1. Let $\eta^{(0)} = 1$. The search interval is $[0, 1]$.

Step 2. Solving model (23) with $\bar{\eta} = 1$ leads to the trivial solution $(\boldsymbol{v}^{(0)}, \boldsymbol{u}^{(0)}, \widehat{\boldsymbol{\gamma}}^{(0)},\ \widehat{\boldsymbol{\delta}}^{(0)}) = \mathbf{0}_5$, $\eta_{max} = 1$.

Second iteration:

Step 1. $\eta^{(1)} = 0.5$. The search interval is $[0, 0.5]$

Step 2. Solving model (23) with $\bar{\eta} = 0.5$ leads to the trivial solution $(\boldsymbol{v}^{(1)}, \boldsymbol{u}^{(1)}, \widehat{\boldsymbol{\gamma}}^{(1)},\ \widehat{\boldsymbol{\delta}}^{(1)}) = \mathbf{0}_5$, $\eta_{max} = 0.5$.

Third iteration:

Step1. $\eta^{(2)} = 0.25$. The search interval is $[0.25, 0.5]$

Step 2. Solving model (23) with $\bar{\eta} = 0.25$ leads to trivial solution $(\boldsymbol{v}^{(2)}, \boldsymbol{u}^{(2)}, \widehat{\boldsymbol{\gamma}}^{(2)},\ \widehat{\boldsymbol{\delta}}^{(2)}) = 10^{-5} \times (2.793, 3.727, 3.212, 0.713, 0.000)$.

Step 3. $\eta_{min} = 0.25$.

---Insert Table 6 Here---

After 14 iterations, the algorithm is ended according to the stopping criterion in which $\bar{\eta} = 0.33758$ as an acceptable approximation for the optimal objective value of model (22). As a result, $\hat{\gamma}^{(14)} = 8.54 \times 10^{-6}$ and $\hat{\delta}^{(14)} = 0$ indicate that the deposit factor is treated as an output. Since the optimal solution of model (22) leads to the optimal solution of model (18), it enables us to attain the common weights as $(v_{\text{ASST}}^*, v_{\text{EQTY}}^*, u_{\text{LOAN}}^*, \gamma_{\text{DEPO}}^*, \delta_{\text{DEPO}}^*) = (2.951, 1.094, 2.960, 0.854, 0) \times 10^{-5}$ (see the last row of Table 6) as well as to calculate the values of the inputs (ASST, EQTY), output

(LOAN), and dual-role factor (DEP) using the change of variables as presented in Table 7. In view of the common weights and the values of $x^*_{\text{ASST},j}, x^*_{\text{EQTY},j}, y^*_{\text{LOAN},j}, w^*_{\text{DEPO},j}$, we can measure the efficiency scores, membership degrees and ranking of 20 banks as presented in Table 8.

---Insert Table 7 and 8 Here---

For instance, let us consider Bank 1. By preforming the variable substitutions, $x^*_{\text{ASST},1} = 285.5, x^*_{\text{EQTY},1} = 56.7, y^*_{\text{LOAN},1} = 232.7$, and $w^*_{DEPO,1} = 31.2$ and then the efficiency score of Bank 1 is calculated as $e_1 = \frac{u^*_{\text{LOAN}} \times y^*_{\text{LOAN},1} + \gamma^*_{DEPO} \times w^*_{DEPO,1} - \delta^*_{DEPO} \times w_{DEPO,1}}{v^*_{\text{ASST}} \times x^*_{\text{ASST},1} + v^*_{\text{EQTY}} \times x^*_{\text{EQTY},1}}$ =0.79094 where $v^*_{\text{ASST}}, v^*_{\text{EQTY}}, u^*_{\text{LOAN}}, \gamma^*_{\text{DEPO}}$ and δ^*_{DEPO} are common weights. In addition, the membership degrees of Bank 1 is $\mu_{\tilde{e}_1}(e_1) = \frac{e_1 - \bar{e}^l_1}{e^u_1 - \bar{e}^l_1} = 0.33812$ where $\bar{e}^l_1$ and e^u_1 are 0.68414 and 1, respectively (see equation (17)). Figure 2 graphically shows the efficiency score e_1 and membership degree $\mu_{\tilde{e}_1}(e_1)$ of Bank 1. Obviously, the incremental increase in the membership degree leads to increase in the efficiency score, for instance, for Bank 1 $[\tilde{e}_1]_{0.5} = [0.84207, 1]$ where $[.]_\alpha$ represents the α-cut of a fuzzy goal[4] (see the red solid line in Figure 2).

As can be observed, the efficiency score of each bank is strongly correlated with its membership degree. That is why the ranking of the banks can be implemented based on either their efficiency scores or membership degrees. The best performance across the available banks is assigned to Banks 3 and 20 concurrently with the efficiency value and membership degree of 1, followed by Banks 14, 11 and 15 with the 3rd, 4th and 5th ranks, respectively, whereas Bank 8 with the lowest efficiency value (0.45698) and membership degree (0.33762) is designated the worst performance.

Contrary to the first four precise scenarios, the proposed algorithm not only reveals the almost identical results regarding the best and worst banks but also it

[4] The α-cut of a fuzzy set A is define to be a crisp subset of universal set X as $A_\alpha = \{X | \mu_A(x) \geq \alpha\}$ where $\mu_A(x)$ is the membership function of A and α varies within [0,1] (Klir and Yuan 1995).

significantly improves the discrimination of the efficient banks in the precise scenarios. Interestingly, the results derived from the algorithm (see Table 8) in line with the results from the two proposed models (see Table 4) where Banks 20 and 3 are the most efficient and Bank 8 is the most inefficient according to the lower bound of the interval efficiencies.

In terms of managerial implications, the methodology in this study provides clear insights as to how ''deposit'' as a dual-role factor can be treated in efficiency analysis of the uncertain banking and finance industry. Specifically, the model settles whether in a bank the deposit factor is behaving as an input, output or it is in equilibrium (c.f. Table 4) along with developing properly model structures for reallocation of the dual-role factor (deposit) across 20 banks in GCC. If it is the interest of the manager to determine the unique role for ''deposit'' among all the banks in addition to monitoring efficiency-based performance, the developed fuzzy decision-making has been employed and consequently the deposit factor is designated as an output. It should be finally emphasized that our analysis in this section is not aimed to demonstrate an in-depth study of the problem, but rather to bespeak the application of the proposed methodology.

5 Conclusion and future research directions

The conventional DEA models undertake the assumption of an explicit designation for each variable identifying whether it is an input or output. There, however, exist some settings where this assumption is not realistic for some variables (so called *dual-roles factors*) whose status is input or output. One of the major drawbacks affecting the application of DEA models in the presence of dual-roles factors concerns uncertainty in the data, particularly public data that tend to be dated and may not reveal present technologies. To deal effectively and constructively with data uncertainty in those DEA models which comes from the lack of available data and data inaccuracy, we propose an interval DEA approach for the purpose of providing a reliable situation for decision makers against ignoring effects of uncertainty. To this end, we first propose a pair of interval DEA models from the pessimistic and optimistic standpoints to measure the interval efficiencies where some or all observed inputs, outputs and dual-role factors are represented by intervals. The models assign the optimal multipliers to each dual-role factor in order to designate that it is an output, input, or is in equilibrium. The consideration of two models from the pessimistic and optimistic viewpoints may reveal two different roles for a given dual-role factor which are frequently not acceptable from the decision makers. Therefore, by taking the interval efficiency score into account as the fuzzy goal for each DMU, we implement the fuzzy decision-making approach based upon the fuzzy max-min criterion. Due to the non-linear model, we develop an algorithm to find the optimal solution so as to obtain the efficiency of each DMU and associated membership degree besides the unique designation of the dual-role factors. We consider a case study of the banking industry to illustrate the applicability of the proposed model. The approach proposed in this research study provides a common framework for future research and we strongly feel that this area would be both economically and methodologically a fruitful area for new research. We particularly propose future research studies on new algorithms for solving the proposed MILP and NLP models by a special focus on improving both execution time and quality of the solutions. Furthermore, the big-M method is leveraged in the developed formulations but in practice a value of M must be defined for computer calculation. A very small value of M may make the solutions erroneous on the one hand and a very high value of M can result in the rounding errors on the other hand. Since finding the proper value of M is beyond the scope of this study, we propose it as another topic for future research.

Acknowledgements
The first author (Mehdi Toloo) was supported by the European Social Fund within the project CZ.1.07/2.3.00/20.0296. The study was also supported by the Czech Science Foundation through project No. 16-17810S. All support is greatly acknowledged.

Appendix: Mathematical Programming code

The GAMS code associated with this article can be found in the online version.

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FIGURES & TABLES

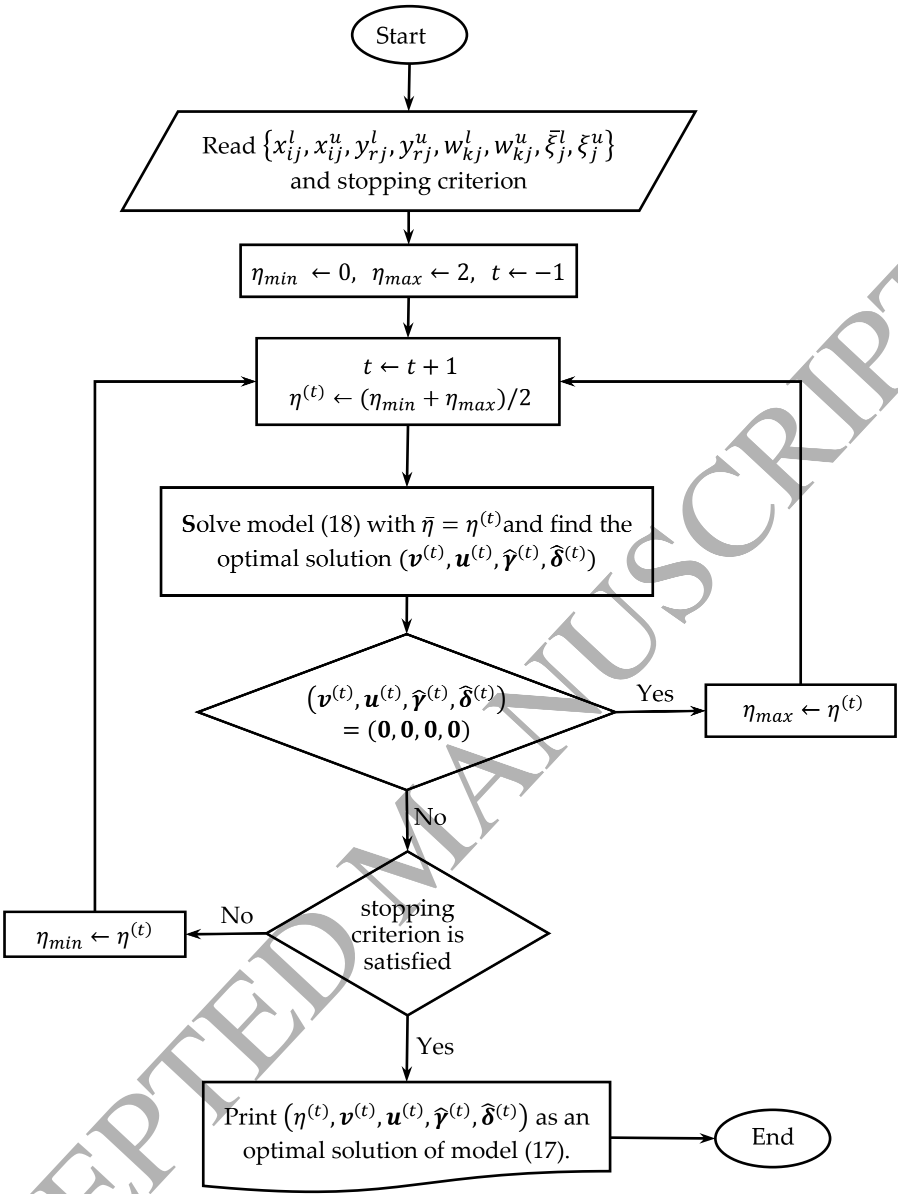

Figure 1. The proposed algorithm

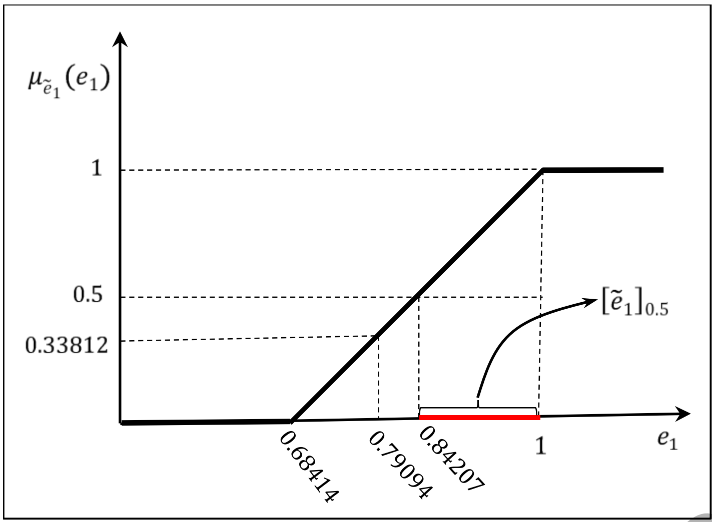

Figure 2 Membership function of $\tilde{\boldsymbol{e}}_{\mathbf{1}}$

Table 1 Input, output and dual-role factor data for 20 banks

DMU	Inputs		Dual-role factor	Output
	ASST	EQTY	DEPO	LOAN
Bank1	[285.5, 299.3]	[56.7, 64.9]	[29.5, 31.2]	[214.2, 232.7]
Bank2	[8522.7, 9652.2]	[1353.9, 1450.5]	[6678.6,7043.1]	[5357, 6228.8]
Bank3	[2443.6, 2494]	[401.1, 439.1]	[1601.1, 1615]	[2187.2, 2556.2]
Bank4	[4953, 5352.9]	[639.9, 648.7]	[4219.3, 4383.5]	[2196.7, 2276]
Bank5	[669.5, 706.1]	[78.4, 91.6]	[579.1, 584.2]	[266.2, 309.7]
Bank6	[5806.6, 6627.4]	[939.1, 1074.5]	[4318.8, 4852.2]	[2451.5, 2470]
Bank7	[17844.5, 20466.3]	[2064.1, 2281.1]	[13733.7, 14569.9]	[14457, 16633.9]
Bank8	[1785.1, 1988.5]	[388.6, 450.7]	[1353, 1376.4]	[438.3, 481.8]
Bank9	[33141.3, 34728]	[1996.6, 2234.4]	[26183.6, 28367]	[17387.5, 17945.4]
Bank10	[13372.5, 14523.3]	[1071.1, 1136.6]	[11533.6, 12718]	[4834.7, 4978.5]
Bank11	[630.7, 687.8]	[115.1, 118.7]	[507.5, 531.1]	[495.5, 529.3]
Bank12	[654.5, 680.7]	[83.5, 89.3]	[554.8, 598.9]	[334.3, 374.7]
Bank13	[1718.3, 1866.8]	[226.7, 237.1]	[1464.3, 1550.3]	[714.9, 783]
Bank14	[1011.5, 1113.7]	[139.4, 141.8]	[835.5, 867.7]	[819.2, 892.7]
Bank15	[4535.5, 5209.8]	[450.9, 467.8]	[3788.3, 4074.1]	[3604.5, 3777.3]
Bank16	[3652.4, 4066.2]	[394.1, 407.3]	[2960.4, 3244.5]	[1614.4, 1845.2]
Bank17	[5356.4, 6311.2]	[575.2, 609.4]	[4688.2, 4789.6]	[2298.9, 2589.8]
Bank18	[684, 731.2]	[129.3, 138.7]	[547.9, 607.8]	[408.6, 463.3]
Bank19	[2702.5, 3074.7]	[281.7, 289.6]	[2268.7, 2623]	[2008.2, 2362.5]
Bank20	[1007.2, 1070.6]	[75, 78.3]	[848.3, 975]	[833.4, 885.9]

Table 2 Efficiency scores and ranking order of four scenarios

DMUs	Scenario 1		Scenario 2		Scenario 3		Scenario 4	
	Eff.	Rank	Eff.	Rank	Eff.	Rank	Eff.	Rank
Bank1	1.00000	1	1.00000	1	1.00000	1	1.00000	1
Bank2	0.92244	18	0.85144	19	0.86171	18	0.80124	19
Bank3	1.00000	1	1.00000	1	1.00000	1	1.00000	1
Bank4	0.97328	14	0.91184	16	0.96605	8	0.89920	13
Bank5	0.98826	10	0.89904	17	1.00000	1	0.90849	11
Bank6	0.85110	20	0.86096	18	0.79832	20	0.80393	18
Bank7	0.99260	9	0.97895	7	0.88320	17	0.86913	16
Bank8	0.86597	19	0.79442	20	0.82963	19	0.76005	20
Bank9	1.00000	1	1.00000	1	1.00000	1	1.00000	1
Bank10	1.00000	1	0.98198	6	0.99908	7	0.96156	6
Bank11	0.95529	15	0.92265	12	0.93080	11	0.90182	12
Bank12	0.97798	12	0.94278	9	1.00000	1	0.96610	5
Bank13	0.97364	13	0.92957	11	0.95816	9	0.91189	10
Bank14	0.99296	8	0.95958	8	0.94666	10	0.91768	8
Bank15	0.98812	11	0.93979	10	0.91562	15	0.87962	14
Bank16	0.93129	17	0.91524	15	0.90400	16	0.87616	15
Bank17	1.00000	1	0.92128	13	0.92454	14	0.83332	17
Bank18	0.93657	16	0.91553	14	0.92928	12	0.91274	9
Bank19	0.99650	7	1.00000	1	0.92458	13	0.93674	7
Bank20	1.00000	1	1.00000	1	1.00000	1	1.00000	1
Mean	0.96730		0.93625		0.93858		0.90698	
Min	0.85110		0.79442		0.79832		0.76005	
Max	1.00000		1.00000		1.00000		1.00000	

Table 3 The optimal values of δ^*_{DEPO} and γ^*_{DEPO} for 20 banks

DMUs	Scenario 1		Scenario 2		Scenario 3		Scenario 4	
	δ^*_{DEPO}	γ^*_{DEPO}	δ^*_{DEPO}	γ^*_{DEPO}	δ^*_{DEPO}	γ^*_{DEPO}	δ^*_{DEPO}	γ^*_{DEPO}
Bank1	0	1.90E-03	0	1.91E-03	0	1.44E-03	0	1.45E-03
Bank2	1.28E-04	0	1.21E-04	0	1.21E-04	0	1.14E-04	0
Bank3	0	3.91E-03	0	4.34E-03	0	2.72E-03	0	3.15E-03
Bank4	2.31E-04	0	2.08E-04	0	2.29E-04	0	2.05E-04	0
Bank5	1.71E-03	0	1.54E-03	0	1.73E-03	0	1.56E-03	0
Bank6	1.88E-04	0	1.77E-04	0	1.80E-04	0	1.66E-04	0
Bank7	3.06E-05	0	2.16E-05	0	2.47E-05	0	1.63E-05	0
Bank8	6.40E-04	0	5.77E-04	0	6.13E-04	0	5.52E-04	0
Bank9	3.82E-05	0	3.53E-05	0	3.82E-05	0	3.35E-05	0
Bank10	8.67E-05	0	7.72E-05	0	8.66E-05	0	7.56E-05	0
Bank11	1.73E-03	0	6.12E-04	0	1.70E-03	0	4.86E-04	0
Bank12	1.67E-03	0	1.57E-03	0	1.72E-03	0	1.61E-03	0
Bank13	6.65E-04	0	6.00E-04	0	6.40E-04	0	5.88E-04	0
Bank14	5.40E-04	0	3.82E-04	0	1.05E-03	0	3.00E-04	0
Bank15	2.41E-04	0	8.51E-05	0	2.25E-04	0	6.42E-05	0
Bank16	2.99E-04	0	2.82E-04	0	3.06E-04	0	2.70E-04	0
Bank17	2.04E-04	0	1.92E-04	0	1.97E-04	0	1.74E-04	0
Bank18	1.60E-03	0	1.51E-03	0	1.60E-03	0	1.50E-03	0
Bank19	4.04E-04	0	3.81E-04	0	3.81E-04	0	3.57E-04	0
Bank20	0	2.55E-03	0	2.06E-03	0	3.03E-03	0	2.61E-03

Table 4 Lower and upper bounds of efficiencies of banks

DMU	$\bar{e}_j^{l*}$	b_{DEPO}^*	d_{DEPO}^*	e_j^{l*}	b_{DEPO}^*	d_{DEPO}^*	e_j^{u*}	b_{DEPO}^*	d_{DEPO}^*	$\bar{\xi}_j^{l*}$	ξ_j^{u*}	b_{DEPO}^*	d_{DEPO}^*
Bank1	0.68414	0	0	1	1	0	1	0	1	0.68414	1	0	0
Bank2	0.54394	0	0	0.71290	0	1	1	1	0	0.54394	0.84932	0	0
Bank3	0.94081	0	1	0.95769	0	1	1	0	0	0.94081	1	0	0
Bank4	0.42468	0	0	0.81212	0	1	1	1	0	0.81212	1	1	0
Bank5	0.38384	0	0	0.84500	0	1	1	1	0	0.38384	0.56376	0	0
Bank6	0.35488	0	0	0.67141	0	1	1	1	0	0.67141	1	1	0
Bank7	0.74493	0	1	0.76203	0	1	1	0	0	0.76203	1	1	0
Bank8	0.21071	0	0	0.70103	0	1	0.94014	1	0	0.70103	0.94014	1	0
Bank9	0.65880	0	0	0.90142	0	1	1	1	0	0.65880	0.84444	0	0
Bank10	0.37546	0	0	0.82010	0	1	1	1	0	0.82010	1	1	0
Bank11	0.68868	0	0	0.79782	0	1	1	1	0	0.68868	0.95695	0	0
Bank12	0.49865	0	0	0.83975	0	1	1	1	0	0.49865	0.68929	0	0
Bank13	0.39192	0	0	0.80816	0	1	1	1	0	0.80816	1	1	0
Bank14	0.75233	0	0	0.81337	0	1	1	0	0	0.75233	1	0	0
Bank15	0.76195	0	1	0.77336	0	1	1	1	0	0.77336	1	1	0
Bank16	0.42816	0	0	0.75039	0	1	1	1	0	0.75039	1	1	0
Bank17	0.39568	0	1	0.76587	0	1	1	1	0	0.76587	1	1	0
Bank18	0.53419	0	0	0.77203	0	1	1	1	0	0.53419	0.77235	0	0
Bank19	0.71288	0	0	0.76223	0	1	1	1	0	0.71288	1	0	0
Bank20	1	0	0	1	0	1	1	0	0	1.00000	1	1	0

Table 5 Reallocation of deposit

DMU	C.D.	C.E.	N.D.	N.E.	Ratio	D.C.
Bank1	[29.50, 31.20]	[0.68414, 1.00000]	[29.50, 31.20]	[0.68414, 1.00000]	1	0
Bank2	[6678.60, 7043.10]	[0.54394, 0.84932]	[6678.60, 7043.10]	[0.54394, 0.84932]	1	0
Bank3	[1601.10, 1615.00]	[0.94081, 1.00000]	[1601.10, 1615.00]	[0.94081, 1.00000]	1	0
Bank4	[4219.30, 4383.50]	[0.81212, 1.00000]	[4219.30, **4641.23**]	[0.81212, 1.00000]	1	257.73
Bank5	[579.10, 584.20]	[0.38384, 0.56376]	[579.10, 584.20]	[0.38384, 0.56376]	1	0
Bank6	[4318.80, 4852.20]	[0.67141, 1.00000]	[4318.80, **4750.68**]	[0.67141, 1.00000]	1	-101.52
Bank7	[13733.70, 14569.90]	[0.76203, 1.00000]	[13733.70, **13733.70**]	[**0.77167**, 1.00000]	1.013	-836.2
Bank8	[1353.00, 1376.40]	[0.70103, 0.94014]	[1353.00, **1488.30**]	[0.70103, **1.00000**]	1.064	111.9
Bank9	[26183.60, 28367.00]	[0.65880, 0.84444]	[26183.60, 28367.00]	[0.65880, 0.84444]	1	0
Bank10	[11533.60, 12718.00]	[0.82010, 1.00000]	[11533.60, **12686.96**]	[**0.87956**, 1.00000]	1.073	-31.04
Bank11	[507.50, 531.10]	[0.68868, 0.95695]	[507.50, 531.10]	[0.68868, 0.95695]	1	0
Bank12	[554.80, 598.90]	[0.49865, 0.68929]	[554.80, 598.90]	[0.49865, 0.68929]	1	0
Bank13	[1464.30, 1550.30]	[0.80816, 1.00000]	[1464.30, **1610.73**]	[0.80816, 1.00000]	1	60.43
Bank14	[835.50, 867.70]	[0.75233, 1.00000]	[835.50, 867.70]	[0.75233, 1.00000]	1	0
Bank15	[3788.30, 4074.10]	[0.77336, 1.00000]	[3788.30, **4167.13**]	[**0.79854**, 1.00000]	1.033	93.03
Bank16	[2960.40, 3244.50]	[0.75039, 1.00000]	[2960.40, **3256.44**]	[**0.75297**, 1.00000]	1.003	11.94
Bank17	[4688.20, 4789.60]	[0.76587, 1.00000]	[4688.20, **5157.02**]	[**0.77087**, 1.00000]	1.007	367.42
Bank18	[547.90, 607.80]	[0.53419, 0.77235]	[547.90, 607.80]	[0.53419, 0.77235]	1	0
Bank19	[2268.70, 2623.00]	[0.71288, 1.00000]	[2268.70, 2623.00]	[0.71288, 1.00000]	1	0
Bank20	[848.30, 975.00]	[1.00000, 1.00000]	[848.30, **848.30**]	[1.00000, 1.00000]	1	-126.7

Note: C.D.= Current deposit; C.E.=Current efficiencies; N.D.=New deposit; N.E.=New efficiencies; D. C.=Deposit change

Table 6 Results of the proposed algorithm

t	η_{min}	η_{max}	$\bar{\eta} = \eta^{(t)}$	$v_1^{(t)}$	$v_2^{(t)}$	$u_1^{(t)}$	$\hat{\gamma}^{(t)}$	$\hat{\delta}^{(t)}$	$z^{(t)}$	$\lvert\eta^{(t)} - \eta^{(t-1)}\rvert \leq 10^{-4}$
0	0	2	1	0	0	0	0	0	0	No
1	0	1	0.5	0	0	0	0	0	0	No
2	0	0.5	0.25	2.793	3.727	3.212	0.713	0	3.7953	No
3	0.25	0.5	0.375	0	0	0	0	0	0	No
4	0.25	0.375	0.3125	2.891	2.092	3.065	0.831	0	3.7136	No
5	0.3125	0.375	0.34375	0	0	0	0	0	0	No
6	0.3125	0.34375	0.32812	2.914	1.719	3.031	0.858	0	3.6949	No
7	0.32812	0.34375	0.33594	2.942	1.245	2.978	0.857	0	3.6686	No
8	0.33594	0.34375	0.33984	0	0	0	0	0	0	No
9	0.33594	0.33984	0.33789	0	0	0	0	0	0	No
10	0.33594	0.33789	0.33691	2.948	1.155	2.967	0.855	0	3.6625	No
11	0.33691	0.33789	0.33740	2.950	1.110	2.962	0.855	0	3.6594	No
12	0.33740	0.33789	0.33765	0	0	0	0	0	0	No
13	0.33740	0.33765	0.33752	2.951	1.099	2.960	0.854	0	3.6587	No
14	0.33752	0.33765	0.33759	2.951	1.094	2.960	0.854	0	3.6583	Yes

Note: The values apropos of the 5th-9th columns must be multiplied by 10^{-5}.

Table 7 The values of inputs, output and dual-role factor for 20 banks by utilizing optimal solution of model (22)

DMU	Inputs		Dual-role factor	Output
	ASST	**EQTY**	**DEPO**	**LOAN**
Bank1	285.5	56.7	31.2	232.7
Bank2	9652.2	1450.5	6678.6	5357
Bank3	2494	439.1	1601.1	2187.2
Bank4	5352.9	648.7	4349.42	2196.7
Bank5	706.1	91.6	579.1	269.63
Bank6	6379.41	1074.5	4852.2	2470
Bank7	20466.3	2281.1	13733.7	14457
Bank8	1785.1	388.6	1376.4	481.8
Bank9	33141.3	1996.6	28367	17963.08
Bank10	14372.21	1136.6	12718	4978.5
Bank11	687.8	118.7	507.5	495.5
Bank12	680.7	89.3	554.8	334.3
Bank13	1866.8	237.1	1556.25	714.9
Bank14	1113.7	141.8	835.5	819.2
Bank15	5209.8	467.8	3788.3	3604.5
Bank16	4066.2	407.3	3244.5	1845.2
Bank17	6311.2	609.4	4688.2	2556.14
Bank18	731.2	138.7	547.9	408.6
Bank19	3074.7	289.6	2268.7	2008.2
Bank20	1070.6	78.3	911.75	833.4

Table 8 Efficiency scores, membership degrees and ranking of the banks

DMU	e_j	$\mu_{\tilde{e}_j}(e_j)$	Rank	DMU	e_j	$\mu_{\tilde{e}_j}(e_j)$	Rank
Bank1	0.79094	0.33812	8	Bank11	0.87985	0.61406	4
Bank2	0.71699	0.37945	11	Bank12	0.69469	0.39103	12
Bank3	1.00000	1.00000	1	Bank13	0.59725	0.33767	16
Bank4	0.61896	0.33769	14	Bank14	0.91187	0.64417	3
Bank5	0.59190	0.33767	17	Bank15	0.87528	0.47606	5
Bank6	0.57272	0.33767	19	Bank16	0.66152	0.40809	13
Bank7	0.86691	0.47822	6	Bank17	0.59976	0.33769	15
Bank8	0.45698	0.33762	20	Bank18	0.72629	0.41239	10
Bank9	0.77408	0.33788	9	Bank19	0.83935	0.44049	7
Bank10	0.58635	0.33767	18	Bank20	1.00000	1.00000	1